education sciences

MDPI

Article

Walking through Algebraic Thinking with Theme-Based (Mobile) Math Trails

Ana Barbosa [1,*], Isabel Vale [1], Simone Jablonski [2] and Matthias Ludwig [2]

[1] Department of Education and Teacher Training, Escola Superior de Educação, Instituto Politécnico de Viana do Castelo, Rua Escola Industrial e Comercial de Nun'Álvares, 4900-347 Viana do Castelo, Portugal; isabel.vale@ese.ipvc.pt

[2] Department of Mathematics and Computer Science Education, Goethe University Frankfurt, 60325 Frankfurt, Germany; jablonsk@math.uni-frankfurt.de (S.J.); ludwig@math.uni-frankfurt.de (M.L.)

* Correspondence: anabarbosa@ese.ipvc.pt

Abstract: Tasks are a key resource in the process of teaching and learning mathematics, which is why task design continues to be one of the main research issues in mathematics education. Different settings can influence the principles underlying the formulation of tasks, and so does the outdoor context. Specifically, a math trail can be a privileged context, known to promote positive attitudes and additional engagement for the learning of mathematics, confronting students with a sequence of real-life tasks, related to a particular mathematical theme. Recently, mobile devices and apps, i.e., MathCityMap, have been recognized as an important resource to facilitate the extension of the classroom to the outdoors. The study reported in this paper intends to identify the principles of design for mobile theme-based math trails (TBT) that result in rich learning experiences in early algebraic thinking. A designed-based research is used, through a qualitative approach, to develop and refine design principles for TBT about Sequences and Patterns. The iterative approach is described by cycles with the intervention of the researchers, pre-service and in-service teachers and students of the targeted school levels. The results are discussed taking into account previous research and data collected along the cycles, conducing to the development of general design principles for TBT tasks.

Keywords: task design; algebraic thinking; math trails; outdoors; MathCityMap; STEM education

check for updates

Citation: Barbosa, A.; Vale, I.; Jablonski, S.; Ludwig, M. Walking through Algebraic Thinking with Theme-Based (Mobile) Math Trails. *Educ. Sci.* **2022**, *12*, 346. https://doi.org/10.3390/educsci12050346

Academic Editor: Ángel Alberto Magreñán

Received: 10 April 2022
Accepted: 11 May 2022
Published: 16 May 2022

1. Introduction

School mathematics requires effective teaching that engages students in meaningful learning through individual and collaborative experiences, giving them opportunities to communicate, reason, be creative, think critically, solve problems, make decisions, and make sense of mathematical ideas [1,2]. In this context, we must stress the importance of complementing formal learning with other environments, like the outdoors, so, according to Kenderov et al. [3], the classroom is just one of the "homes" where education takes place. Learning mathematics is a lot about discovering, using and understanding mathematical contents in and through the students' daily environment and, in this sense, the idea of Outdoor Mathematics has gained popularity. A possible approach is the use of mathematics trails (in the following, math trails). A math trail guides students along a route with mathematical tasks related to real objects, like a handrail, a stone or a staircase.

Recent research shows that math trails can have positive impacts on learning outcomes if used regularly [4] and might emphasize mathematical modeling in an enriching way [5]. The affective environment can influence the initial expectations and motivations of students for the study of mathematics, and the use of the surroundings, as an educational context, has the potential to act as such an environment, allowing them to understand the applicability of mathematics [6]. Students need understanding and the establishment of connections are fundamental to achieve this goal. Still, teachers show concerns about the use of math trails, because of the necessary preparations and missing ideas for tasks that suit the particular curriculum [7]. It seems that math trails play a minor role in daily teaching rather than for specific purposes, like excursion days.

In the strategic partnership MaSCE[3] (Math Trails in School, Curriculum and Educational Environments of Europe; 2019–2022), the gap between the math trails' potentials and their actual use in schools should be closed. On a content-based level, theme-based trails (TBT) are developed to facilitate the implementation of math trails and to show the relevance of the math trails' tasks for the national curricula as well. A TBT includes several tasks that are linked to real objects outside the classroom and that can frequently be found close to schools. All these tasks focus on a particular topic from the curriculum and cover different sub branches of this topic.

In order to create a successful TBT in the context of outdoor mathematics, the tasks and the trail have to meet certain criteria and standards. Since the idea of TBT in the outdoor context is rather new, this study aims at identifying the principles of design for mobile TBT that result in rich learning experiences in early algebraic thinking, a topic chosen by the researchers. To do so, the paper starts with a theoretical framework concerning task design and the outdoor context. Again, the idea of math trails is presented, but in a more detailed way by reflecting the use of digital tools. Afterwards, the methodology for setting up the TBT task design principles is described. It takes advantage of the theoretical framework, discussion of experts and empirical field testing with pre-service and in-service teachers as well as students. Taking these aspects into consideration, we present the resulting design principles as a guideline for future TBT and task design principles in different contexts.

2. Theoretical Framework

2.1. Task Design in Mathematics Education

Tasks are the basis of the elaboration of a trail, so it is pertinent to discuss some aspects related to task design. We assume that the students' learning progress is influenced by the tasks proposed, requiring them to think conceptually, to make connections and be motivated [8–11]. From this perspective, it is important to create "good" mathematical tasks. Following curricular guidelines, a task is labeled "good" if it allows the introduction of fundamental mathematical ideas, if it is an intellectual challenge for students and facilitates the use of various approaches, and if it is based on sound and significant mathematics; knowledge of students' understandings, interests, and experiences; and knowledge of the range of ways that different students learn mathematics [12]. The expression 'mathematical tasks' can adopt different meanings. For Stein and Smith [10], a task is a statement proposed by the teacher in the classroom with the goal of focusing students' attention on a certain mathematical idea or content, with different levels of cognitive demand, and which implies an activity on the part of the student. On the other hand, according to NCTM [1], a task includes both the mathematics and the varying levels of cognitive demand or processes needed to solve it.

All students should have the opportunity to engage in a meaningful mathematical activity and it is the teacher's role to unlock their potential through a choice of adequate tasks and teaching strategies. Although tasks have the power to trigger mathematical activity, they may not be sufficient to implicate mathematical challenges. Teachers must establish a classroom environment that guarantees students' engagement in embracing mathematical challenges [11]. Challenge is an important variable in mathematics learning, because students can become unmotivated and bored very easily in a routine class. Some may even have difficulties in learning, unless they are challenged [13]. The expression 'challenging task' is normally used to describe a task that is interesting and perhaps enjoyable, but not always easy to deal with or to attain, and should actively engage students, developing a diversity of thinking and learning styles. The challenge is usually linked to problem solving, and generally, when researchers use the term 'problem solving', they are referring to mathematical tasks that have the potential to provide intellectual challenges that can improve students' mathematical development [12]. This challenge can appear through the proposed tasks or through scenarios where the process of teaching and learning occurs and also through the strategies used by teachers to develop that process. According to Hartmann and Schukajlow [14], working outside the classroom on tasks of a real-world context is more motivating than working with photographs, videos or books in the classroom. A possible explanation could be that the outdoor learning environment offers an authentic experience with on-site objects or phenomena, and hence students conclude that solving that task is more valuable and meaningful. Experiencing a math trail, students can solve mathematical tasks related to real objects or phenomena. The task context is not just to motivate students, but also to provide them with a real learning situation that can be used as a starting point for a sound understanding.

Different tasks with different levels of cognitive demand induce different modes of learning. Consequently, there are tasks of a high or low cognitive demand, as they give students more or less opportunities to engage in complex thought processes. Smith and Stein [9] characterize the level of cognitive demand of a task in four levels based on the kind of thinking required to solve it (Table 1):

Table 1. Levels of cognitive demand of tasks [9].

Lower-Level Demand	Higher-Level Demand
(1) *Memorization* Involves the reproduction of previously memorized facts as rules, formulas, facts or definitions.	(3) *Procedures with connections* Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.
(2) *Procedures without connections* Involves the use of algorithmic procedures that are evident in the task statement.	(4) *Doing mathematics* Requires complex thinking instead of algorithmic procedures and strategies in solving the tasks. Students have to access relevant knowledge and experiences and make appropriate use of them in working through a task.

Higher-level tasks should be used instead of lower-level tasks, as these involve performing many similar steps, becoming routine work, with evident excessive use of memorization. More recently, Mullis and Martin [15] proposed another characterization of the mathematical tasks, with different cognitive domains, known as TIMSS cognitive domains, classified into three levels (Table 2):

Table 2. Mathematics cognitive skills domains [15].

Knowing	Applying	Reasoning
Imply the evocation and repetition of knowledge that has already been taught, cover the facts, concepts, and procedures students need to know.	Requires the integration and relationship of diverse mathematical knowledge, based on knowing and framed in non-routine situations, related to familiar settings, focuses on the ability of students to apply knowledge and conceptual understanding to solve problems or answer questions.	Demand reasoning and reflection to achieve the solution and to go beyond the solution of routine problems to encompass unfamiliar situations, complex contexts, and multi-step problems.

As we consider these two frameworks of task demand to be similar, we chose to adopt the one proposed by Smith and Stein [9] to support the creation of the tasks that compose the trails in this work.

Another aspect to consider relating to the tasks is the context, which, in the scope of this study, needs to be highlighted because the tasks are carried out outside the classroom, in a real-life context. The role of contexts in tasks is a complex issue that goes far beyond simply motivating students to tackle a task, as already suggested. The importance attributed to a task's context is more associated with the opportunity represented for mathematical abstraction through different situations and different representations, rather than making the mathematical context familiar to students. However, the context has an important role because it is a way of seeing the applicability of mathematics, and also of making mathematical knowledge more accessible to students. More than this, the potential of a context task to generate discussion and abstraction depends on how it is handled [16]. For Borasi [17], one of the main roles of the context for carrying out a task is to provide the solver with information that may allow him/her to find the solution. We use the term context, in the sense proposed by van Heuvel-Panhuizen [18], as a learning environment that includes the different situations in which learning takes place.

When we use the environment as the context of a math trail, we need to know how to create the different mathematical tasks that compose the trail. Among the different tasks that we use in mathematics learning, problem solving plays an important role in developing the learners' skills, involving rich discussions that are considered cognitively challenging, and are the primary mechanism for promoting a conceptual understanding of mathematics. In addition to these aspects, problem posing is a powerful strategy to develop problem solving skills and to gain good problem solvers. Silver [19] considers problem posing either being the generation (creation) of a new problem or the reformulation of a given problem. Stoyanova [20] considers problem posing as the process by which, on the basis of mathematical experience, we construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems. The process of the creation (invention, formulation, elaboration) of problems can be defined in several ways, but, in essence, the previous authors describe equivalent activities. In both definitions, the term 'problem' is used for any kind of mathematical task, whether it is a routine or a non-routine problem; thus, problem posing can lead to the creation of any kind of task [21].

Mathematics tasks can be created or reformulated in diverse ways depending on the level of the students and the teaching aims. For task design, we will use the ideas of problem posing. From the work of Brown and Walter [22] relating to strategies for problem posing, we identify two that we can use: (1) accepting the given, starting from a static situation (e.g., an image, an object, a phenomena, expression, table, image, sentence, a condition, a diagram, a calculation or simply a set of data) from which we formulate questions in order to find a problem without changing the starting situation; and (2) what-if-not, an extension of a given task by changing what is given. From the information contained in a problem, it is possible to identify what is known, what is requested and what limitations the response to the problem involves. By modifying one or more of these aspects or questions, new and more questions can be generated [2,23]. Still, Sullivan and Liburn [24] propose a practical and accessible method for posing open-ended questions to a specific situation using a three-step process: (1) working backward, which includes identifying a topic, thinking of a closed question and writing down the answer and making up a question that includes (or addresses) the answer; and (2) adapting a standard question, which includes identifying a topic, thinking of a standard question and adapting it to make a good question [2,23]. These methods [22,24] provide us with information relating to the way we choose a mathematical object translated into mathematical photography, and if we look for the math potential of an object (or phenomena) in a photo or if we look for an object that matches a predefined math content. This type of photograph, which we call mathematical photography (or a problem picture), is a photograph of an object, phenomenon or situation that is accompanied by one or more questions, or a problem based on the context of the photograph [25]. Regarding task design, we agree with Richardson [26]: questions should arouse curiosity, forcing the students to observe the environment in order to achieve a successful solution.

Hence, it is important that teachers apply their knowledge concerning the creation of tasks outside the classroom so that they can design tasks (isolated or in a trail) for their own students. Formulating problems helps teachers (and students) to consolidate problem-solving skills and to strengthen their mathematical knowledge and skills. Moreover, by doing this in the surrounding environment, it makes it easier for students to establish mathematical connections between aspects of everyday life, seeing the applicability of mathematics as well as developing their own creativity. In particular, it encourages teachers to ask questions, an essential aspect of the teaching and learning process. Thus, great attention is necessary to support teachers in the construction and refinement of these tasks [2,23].

2.2. A Didactical Sequence to Develop Algebraic Thinking

Many mathematicians have come to recognize the study of patterns as a fundamental part of the curriculum of school mathematics. Some even consider mathematics to be the science of patterns [1,12,27,28]. The search for patterns is seen as a way of approaching algebra, since it is a fundamental step for establishing generalization, which is the essence of mathematics [28–32]. There has been an agreement to defend the integration of algebra from the early years, with an emphasis on experiences that allow students to "learn algebra both as a set of concepts and competencies tied to the representation of quantitative relationships and as a style of mathematical thinking for formalizing patterns, functions, and generalizations" [12] (p. 223).

To be competent at algebra, students must understand concepts and relationships beyond mere symbolic manipulation, which implies that its study begins in the early years with the development of algebraic thinking, where the search for patterns and the establishment of conjectures and generalization in figurative contexts play a crucial role, including the use of different representations [12,28,31,33,34]. Pattern generalization tasks are often used in the classroom to involve students in identifying a pattern, in its extension to finding the value of a near or far term, and then articulating it with the underlying functional relationship using symbols. It can thus be said that there are basal processes in the promotion of algebraic thinking. The existing literature attributes different connotations to the term 'algebraic thinking'. In general terms, we highlight the definition proposed by Blanton and Kaput [33], who consider algebraic thinking to be "a process in which students generalize mathematical ideas from a set of particular cases, establishing these generalizations through the discourse of argumentation, gradually expressing them in formal and age-appropriate ways" (p. 413). Some researchers [34] also emphasize awareness of the existence of patterns and structures as central ideas in algebraic thinking. Regardless of any other definitions used [1,12,31,33], the fundamental principles of algebraic thinking contemplate the construction of generality throughout the curriculum. Algebraic thinking can take different forms, including functional reasoning, which has been identified as one of the main axes of this type of thinking [32,34] and as one of the central contents in algebra research in the early years. It can be defined as the process of thought used in the construction and generalization of patterns and relationships, using different linguistic and representational tools and exploring generalized relationships or functions, which constitute mathematical objects in their own right [35].

In the context of algebra, the tasks of pattern exploration can be proposed in a variety of contexts (e.g., numerical, geometric, figurative), using a visual support that allows the application of strategies of different natures in an attempt to reach generalization, giving meaning to the expression of generality and becoming more understandable for many students [28,29,34,36]. The ability to recognize patterns and reorganize data to represent situations in which the input is related to the output, through well-defined functional rules, is critical for the development of algebraic thinking [1,29,30]. The exploration of patterns thus constitutes a privileged means by which to introduce algebra due to the possibility of a dynamic representation of the variables involved.

Research developed with (future) teachers has shown that their knowledge of algebraic concepts is often fragmented and consists of a system of symbols and procedures with no connection between them [37]. Effective teaching of algebra requires the careful preparation of teachers throughout their training. This implies that they have their own experiences with richer algebra, in which connections are contemplated, and they must understand how they can build opportunities of this nature with their own students [1,33]. Different studies that we have developed on the relevance of using tasks involving the discovery and study of patterns in figurative/visual contexts, as an essential component of the construction of generalization [29,38], show that an approach of this nature allows the motivation of students and teachers for mathematics, develops higher order skills and stimulates creativity, valuing relationships between different mathematical themes in different contexts.

Pattern tasks, involving counting and repetition/growth sequences, facilitate the formulation of conjectures and the expression of generalizations emerging from inductive reasoning, which is accessible to elementary school students. Furthermore, they encourage the establishment of connections between various modes of representation. This connection allows a better understanding of the underlying mathematical structure, leading more effectively to conjecture, generalization and argumentation.

The tasks of the trail about Sequences and Patterns, described in this article, followed a didactical sequence proposed by Vale, Barbosa, Borralho, Barbosa, Cabrita, Fonseca, and Pimentel [38], that defines a learning trajectory, consisting of three fundamental phases for the development of algebraic thinking: Counting; Sequences; and Problems; which is presented in Figure 1.

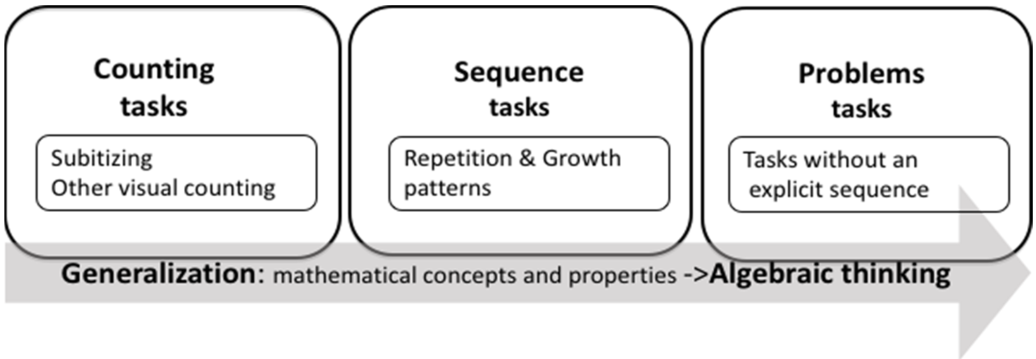

Figure 1. Didactical sequence.

This didactical proposal involving patterns that allow different types of approaches assigns a fundamental role to visualization, favoring figurative contexts and including higher thinking processes that are an essential part of problem solving, but also of algebraic thinking (such as analyze, continue, conjecture, generalize, justify and represent). The sequence of tasks focuses on the development of generalization as a pathway for algebraic thinking. It is also added that relevance was given to the use of tasks that allow different approaches and representations of an analytical and visual nature, in order to promote algebraic thinking and an understanding of the underlying structure. Following this proposal, and to achieve the flexibility of thinking resulting from the use of different ways of seeing, which best suits our purposes, it is necessary to start by exploring basic visual counting tasks to develop, among others, the ability to use subitizing (instantly seeing), evolving to visual counting in different contexts, in order to promote pattern recognition in various dispositions, as a requirement for the posterior phase with sequences. Here we began with figurative/visual sequences followed by numerical sequences, which privilege the visual intuition regarding numbers and their relationships. The next phase contemplates tasks that involve discovering patterns and building generalizations, with patterns of repetition (there is an identifiable unit/motif that repeats itself in a cyclic way indefinitely) and growth (each term changes in a predictable way in relation to the previous one) patterns, aiming at the recognition, discovery, continuing, completing and generalizing of patterns. Although all types of sequences are necessary for the development of mathematical thinking, growth patterns lead more naturally to the discovery of relationships between two variable quantities, facilitating algebraic reasoning [39]. In the last phase, problems were proposed where the underlying sequence is not presented in an explicit way, so students will have to discover, explore and construct their own sequence, establishing generalizations to reach a solution. In this last phase, in addition to repetition and growth patterns, other types of patterns appear, namely those whose discovery leads to invariants that allow the establishment of numerical or geometric properties. At any time, each task may be solved using different representations. Thus, the confrontation of ideas and their justification are fundamental to promote learning and to extend the repertoire of solution strategies.

To design the tasks for the mobile TBT regarding Sequences and Patterns, we followed the strategy 'accepting the given', proposed by Brown and Walter [22], and the two methods for posing good questions, proposed by Sullivan and Liburn [24], based on objects in the real-world context. We also grounded our options in the ideas of other authors [19,20], who embrace all kinds of tasks, from simple exercises to challenging problems, and also consider different levels of cognitive demand [9].

2.3. Outdoor Mathematics in a Digital Context

Mathematics does not have to take place exclusively in the classroom. Especially when it comes to implementing "realistic" mathematics tasks, i.e., tasks that are connected to situations in the real world, so called "outdoor mathematics" seems to be a potentially beneficial learning opportunity. One approach is a mathematics trail (also known as a "math trail"). They describe a route that guides students to mathematics tasks located in the environment and at real, physical objects/phenomena [40]. When running a math trail, these objects become the center of mathematical tasks through appropriate questions. In addition, the tasks of a math trail can only be solved on-site as it is necessary to choose and collect data to solve the problem at these places [41]. Hereby, the students transfer tasks that are known from the textbook to everyday objects, places and concrete situations which requires the acquisition of (new) problem solving strategies [42]. Experiencing a math trail makes them more aware and attentive to the mathematics that surrounds them in a real context, awakening their "mathematical eye", through contact with more meaningful tasks [43]. Originally, math trails were created to popularize mathematics in society. For example, Blane and Clarke [44] created a math trail in Melbourne in the 1980s, intended as a vacation activity for the whole family. Accordingly, the tasks were set in such a way that simple basic arithmetic was sufficient to discover or just discuss mathematical relationships and phenomena in the environment. This idea is still relevant in today's math trail approaches that often transfer math trails into the educational context [45]. Moreover, the MathCityMap (MCM) system (www.mathcitymap.eu, accessed on 12 November 2021) focuses on the use of math trails in the educational and—in addition—the digital context.

The MCM system consists of two main technical components. First, there is a web portal (www.mathcitymap.eu; Figure 2), which is an international database that supports teachers in the creation of their own tasks and math trails. On the other hand, there is a corresponding smartphone and tablet app (Figure 3), which loads a selected math trail from the portal and makes it available to students while they are walking along the trail.

Figure 2. The MCM Web Portal.

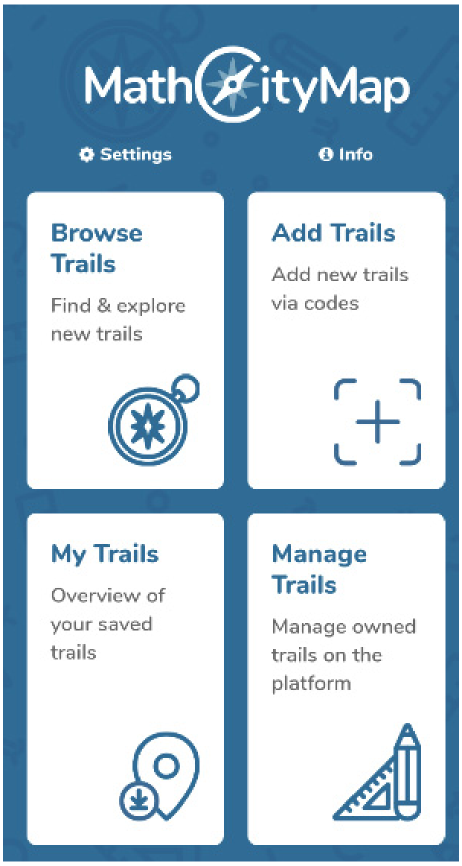

Figure 3. The MCM smartphone/tablet App.

In the following, we focus on the teacher's perspective in terms of the web portal and principles for task creation. For an introduction into the features of the app, see, for example, Ludwig and Jablonski [46]. The web portal offers the possibility to view tasks in order to get ideas and create your own tasks. When creating tasks, the portal allows the user to position the task pin on the map and to upload a photo of the task's object. The information is necessary since the tasks are linked to one individual location and object. During the task creation, it is possible to choose from different answer formats which have been evolving over the project's duration as the result of feedback and ongoing improvements in the system. As a solution format, the system allows the following answers (Table 3):

Table 3. Overview of different answer formats available in the MCM system.

Answer Format	Description	Use Cases
Exact value	The exact value is only one number as the correct answer.	It can be used for tasks in which definitely only one correct answer exists, e.g., for counting tasks or for combinatorial problems.
Interval	By setting up an interval, teachers define a green branch for "good" solutions and an orange branch for "acceptable" solutions. Everything outside the green and orange branches is validated as "wrong".	The interval used whenever measurements are necessary, e.g., to determine a length, an area, or a volume.
Multiple choice	Available data can be queried within the *Multiple Choice* format like in a quiz. Thereby, at least two answer options must be given, of which at least one is correct.	The multiple choice answers are open to every situation. It is especially recommended for recognizing mathematical characteristics.
Fill-in-the-blanks	Within this format, gap texts can be easily worked on outside the classroom.	The fill-in-the-blanks format is useful to analyze objects outdoors in technical language, to deal with data from information boards or to raise questions on data of historical realities.
Set	If several numbers are the expected solution in a task, but the order in which the numbers are to be entered is not important, the *set* task format can be used.	The *set* answer format can be used for tasks in which more than one correct answer exists and all of them can be clearly identified.
Vector	To raise more than one question on a measuring activity, the task format *vector* (*interval)* can be used. Analogously, we offer the *vector (exact value)* format, which can be used to set several counting tasks or combinatorial problems at once.	The vector can be used to check several measurements. Moreover, the task format can be applied for questions concerning spatial geometry.

Depending on the answer format, the app validates the entered solution and gives immediate feedback. Independent from the answer format chosen, a task creation also contains the formulation of a sample solution that can be retrieved after working on the task and the formulation of stepped hints that can be retrieved already during the solving process.

We pointed out that the design of mathematics tasks requires certain criteria to be successfully implemented in the teaching and learning of mathematics. Surely, the outdoor mathematics learning situation in the context of MCM use asks for additional and modified task design criteria, such as the following:

- *Uniqueness*: "To make clear which object is meant, every task should provide a picture that helps identify the object of the task and what the task is about" [47] (p. 118).
- *Attendance*: "A task should be authentic, i.e., leaving the educational context and having a certification. Thus, the task can only be solved at the object location and its description should never be enough to solve it" [47] (p. 118).

- *Activity*: "Physical activity has a positive effect on learning, implying the idea of embodied mathematics, i.e., mathematics can only be fully comprehended through an active experience (Tall, 2013). The task solver should therefore become active and do something in order to solve the task, e.g., measure and count" [47] (p. 118).

To make it easier for teachers to create tasks that fulfill these exemplary outdoor task design criteria, there is a catalog of so-called "Generic Tasks". The intention here is to make frequently found task objects, such as the slope of a ramp or the speed of an escalator, transferable to new locations with as little effort as possible [46].

By connecting several tasks, a math trail can be created. In particular, if the tasks focus on one mathematical topic, a TBT is created: "A theme-based trail is a collection of [...] tasks of one common topic, e.g., fractions in arithmetic, percentage calculation in algebra, linear functions in analysis, as well as combinatorics in stochastic theory. Within a theme-based trail, a specific curriculum topic is addressed and can therefore be directly connected to regular math class" [48] (p. 147). In the following, we will present the development process of a TBT on algebraic thinking.

3. Method, Context and Participants

This study is part of a European project, MaSCE^3—Math Trails in School, Curriculum and Educational Environments in Europe—developed within the scope of the Erasmus+ Program, Key Action 2. One of the intellectual outputs of the project is the design of TBT with MCM, as a collection of tasks focused on a specific mathematics topic, one of them being algebra. The aim of this study is to identify the principles of design for mobile theme-based math trails that result in rich learning experiences in early algebraic thinking, which led us to use a qualitative and interpretive approach [49] in the form of design research methodology [50]. The idea is to generate a theory about the learning process and the means of supporting that learning [51], concerning the development of TBT using MCM. This methodological option implied the implementation of iterative and reflective cycles, first informed by the existing literature about task design, involving the development of algebraic thinking, and outdoor mathematics in a digital context, ultimately resulting in new knowledge (design principles) emerging from reflection on the results of the conducted experiments (cycles). Thus, this study fits one of the archetypes of design research and that is the production of curricular products/design principles, ready to be used by practitioners/teachers and instructional designers [51]. The research results, expressed in the form of design principles, derive from cyclic processes of design, formative evaluation and revision of the initial propositions [52].

The study was carried out using an iterative and process-oriented approach conducted through the collaboration between the researchers (i.e., the MaSCE^3 project partners) and teachers (i.e., Portuguese pre-service and in-service teachers). This design research had a learning trajectory with three cycles, each consisting of the following phases (see Figure 4): (i) preparation of the intervention; (ii) conducting (classroom) experimentation; and (iii) retrospective analysis of the collected data that informed the next cycle [53]. The analysis carried out on each experimentation phase generated information that supported the preparation phase of the next cycle. The first cycle was conducted with nine researchers from the project team, the second cycle involved 24 pre-service teachers from elementary education (future teachers of 6–12 years old students) and the third was implemented with two teachers and their students, 23 from 4th grade and 25 from 6th grade. The inputs from the researchers and teachers involved, based on their perceptions and experiences, were crucial to evaluate and refine the tasks and the trails, and obviously the related design principles.

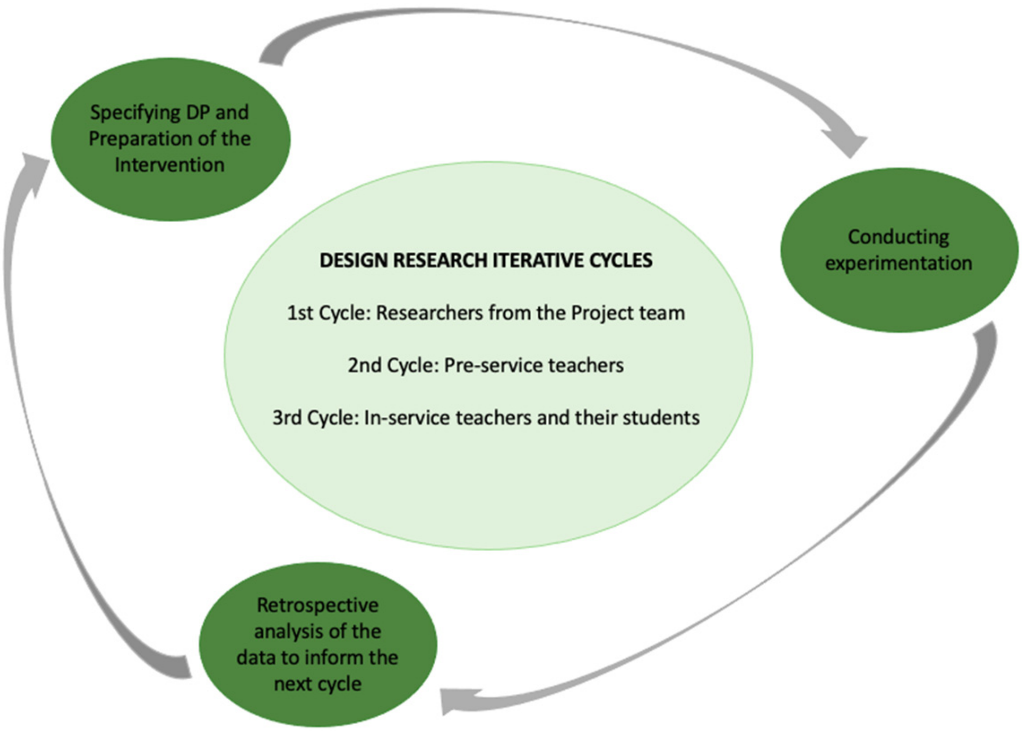

Figure 4. Design Research Iterative Cycles.

The first cycle started with the elaboration of the first version of the TBT in the scope of algebraic thinking in the early years, about Sequences and Patterns, taking into account three main aspects that contributed to the formulation of the preliminary design principles: analysis of the curricular guidelines; a literature review about task design and the development of algebraic reasoning; and answer formats allowed in MCM and quality criteria that influence task design. The TBT was then analyzed by the project partners (Portuguese, German, French, Italian, Spanish) in order to anticipate the adequacy of the proposal through the lens of mathematicians and teacher educators. Data collected through the written feedback and informal conversations in two meetings, that led to field notes, helped to reflect specifically about the tasks, the trail and generally about the design principles. The retrospective analysis of these data helped inform the following cycle, the experimentation of two TBT about Sequences and Patterns (grades 3–4 and 5–6) with pre-service teachers. After revising the preliminary design principles, improving the tasks and the trail, we prepared an intervention with the aim of implementing the TBT with future teachers from elementary education that had previous experience as MCM users, hence focusing completely on the tasks and the trails. At the moment of the study, the 24 participants were attending the second semester of the 1st year of a Master's course, with a two-year duration, qualifying them for the teaching of Mathematics and Sciences for grades 1–6. The researchers supervised the trail implementation, taking photos and observing the participants' comments and reactions, ultimately collecting the tasks' solutions. The future teachers solved all the tasks of the TBT, working in groups of three, and after the trail implementation, they filled in a questionnaire. This instrument was designed to assess the future teachers' perceptions regarding outdoor education, MCM use and ultimately the tasks solved, as well as the TBT about Sequences and Patterns. The questionnaire contained open-ended questions with the intention of getting more in-depth responses, concerning opinions, interpretations and reactions. The design principles were once again revisited after the experimentation phase, leading to the creation of two TBT in two different schools, one for 4th graders and another one for 6th graders. These trails were experimented with students and were supervised by the respective teachers. The researchers accompanied and observed these moments, taking photos, collecting field notes and the tasks' solutions. The two teachers were also interviewed after this experience. The researchers conducted semi-structured interviews, which are useful to instigate a more open conversation about broad, but also specific, topics. These conversations were structured with the expected flexibility of an interview of this nature, trying to understand the teachers' perceptions about outdoor education; MCM use in elementary education; the students' reactions; the nature, adequacy and pertinence of the tasks; and the TBT.

Globally, the experiences conducted along the three cycles served as a basis for revising the design principles concerning the TBT regarding Sequences and Patterns, and particularly the tasks.

4. Results and Discussion

To present and discuss the results, we chose to detail the work developed in each of the three cycles, starting with the definition of the design principles for the TBT tasks, followed by an analysis of the experimentation and its impact on the subsequent cycle.

4.1. 1st Cycle

To start the process, we formulated four design principles (DP), taking into consideration previous research, the existing theoretical framework about task design and mobile outdoor education, and curricular guidelines related to the development of algebraic thinking:

- **DP1**—Formulate a catalog of generic tasks based on objects/phenomena that might be found in schools or in the surrounding environment, to inspire teachers to more easily adapt the proposals to their educational context [46].
- **DP2**—In order to develop algebraic thinking in elementary grades, tasks should contemplate the following concepts: counting in visual contexts (subitizing); combinatorial counting; repetition patterns; growth patterns. This learning trajectory is sustained by research and can be considered as a possible pathway to work on generalization [28].
- **DP3**—Differentiate the tasks' features in order to diversify the level of cognitive demand [9]. Lower-level and higher-level tasks imply different procedures and reasoning, routine and non-routine approaches, which contribute to a more interconnected mathematical understanding.
- **DP4**—Formulate the tasks according to the MCM portal features, namely available answer formats and quality criteria [47].

We started to design tasks, taking these principles into consideration. After analyzing the curricular guidelines for grades 1–6 about algebraic reasoning and choosing the main concepts related to this topic, we followed Vale et al.'s [28] didactical proposal and thought about objects with mathematical potential, that could be found in or near schools, common to the educational contexts of the project's partner countries. Then, we tried to achieve proposals that met the principles of task definition in MCM and diversify the level of cognitive demand. The generated tasks were not organized in the form of a trail. In fact, they were generic tasks, involving the above-mentioned concepts (DP2), that intended to serve as inspiration for teachers to pose their own tasks. As an example, Figures 5 and 6 illustrate two generic tasks about repetition patterns, with different levels of cognitive demand.

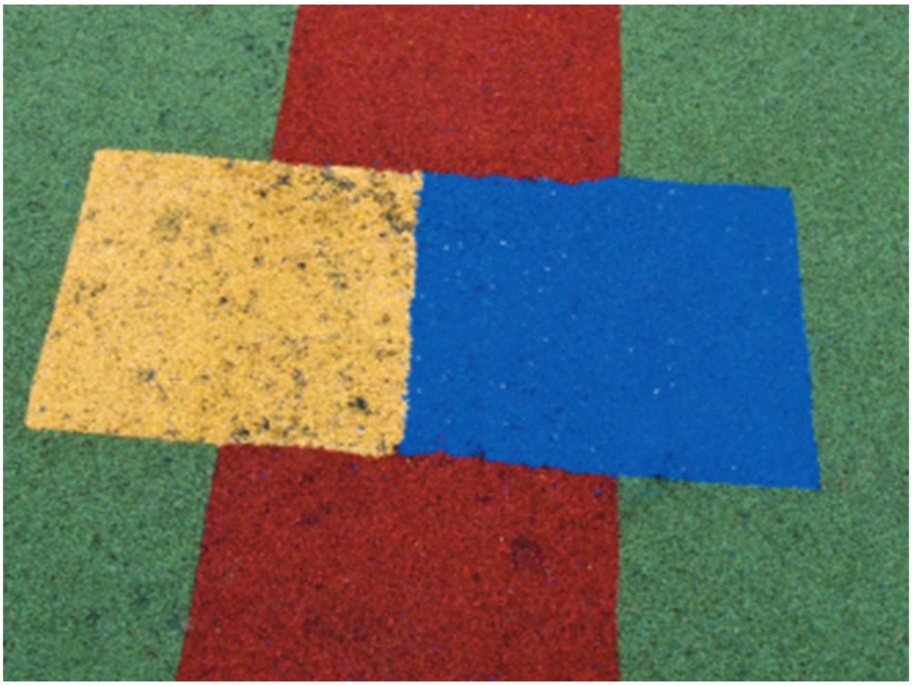

Figure 5. Generic task about repetition patterns (lower-level of cognitive demand).

Figure 6. Generic task about repetition patterns (higher-level of cognitive demand).

As we can see, the tasks focused on objects easily found in schools (e.g., hopscotch, fences) and had a generic statement to give an idea of the underlying goal. These tasks were accompanied by supplementary information to help teachers with the transition to a concrete task: a list of objects to which the task could be adapted (e.g., railing, trees, gate, sequence of repeated objects) and a list of MCM tasks' referring numbers, corresponding to particular tasks already published and available in the MCM web portal (https://masce.eu/the-project/intellectual-outputs/theme-based-trails/theme-based-trails-statistics/theme-based-trails-sequences-and-patterns/, accessed on 12 November 2021).

The experimentation phase had the participation of nine researchers from the project team that analyzed the first version of the TBT, specifically its tasks. Two meetings were held and emails were exchanged, to discuss the adequacy of the proposals and the underlying design principles, leading to the collection of written documents and field notes.

There was an agreement that the tasks did not need to cover the whole curriculum, but rather it was more important to identify the main topics of the theme, algebraic thinking, that could be easily illustrated in MCM. Having this idea in mind, Vale et al.'s [28] didactical proposal was well accepted, which allowed us to maintain DP2, organizing a trail focused on Sequences and Patterns. It was also consensual that DP4 should be considered as a basic principle of the TBT tasks, since the MCM web portal has specific features to obey and the tasks have to satisfy certain criteria (e.g., available answer formats; MCM tasks can only be solved on site). On the other hand, the analysis of the proposed tasks generated some discussion concerning its generic nature and the absence of a trail-like structure. There was a common assumption that the generic tasks may not be as useful to teachers as particular tasks, which led to an agreement about the necessity of organizing a didactical documents with detailed information about the TBT: MCM trail code, grade, main concepts, what students learn, data to collect, objects, and MCM task references. This document would help to guide teachers in developing their own trails, through specific examples of a trail and its tasks, displaying at the same time the learning trajectory. Attention was also drawn to organizational and logistical aspects. In some countries, elementary education students—the target audience of the TBT—may not be allowed to walk unsupervised around the school premises for safety reasons, or even if they were authorized, it can be seen by teachers as a great responsibility or as an activity difficulty to manage them, especially with classes of more than 20 students. To overcome these constraints, the tasks should focus on objects/phenomena placed in the school grounds. Finally, regarding the level of cognitive demand of the tasks, and after some discussion, the group agreed that diversifying the typology of the tasks in this category could bring added value to the TBT; however, it was also considered that there should be some balance in the level of challenge of the tasks, since higher-level tasks normally take more time to solve than routine tasks. Thus, the TBT should have more lower-level tasks than higher-level ones.

To synthesize, the data analysis resulting from this first cycle produced results that were useful to further develop the design principles for the TBT tasks, introducing changes to DP1 and DP3.

4.2. 2nd Cycle

The second cycle of this research started by revisiting the design principles, informed by the retrospective analysis of the data collected in the previous cycle, introducing changes to DP1 and DP3:

- **DP1**—Formulate a set of particular tasks, organized in the form of a math trail, based on specific objects/phenomena that can be found in schools, to inspire teachers to more easily adapt the proposals to their educational context [46].
- **DP3**—Differentiate the tasks' features in order to diversify the level of cognitive demand [9]. Lower-level and higher-level tasks imply different procedures and reasoning, routine and non-routine approaches, which contribute to a more interconnected mathematical understanding. Balance the level of challenge of the tasks, introducing more lower-level tasks than higher-level ones.

Based on these principles, we developed two TBTs on Sequences and Patterns with particular tasks, one for grades 3–4 and another one for grades 5–6. The trails had the main concepts in common (visual counting, combinatorial counting, repetition patterns and growth patterns) but had the proper adaptation to the knowledge expected of each age group. Our intention in this second cycle was to implement the trails with pre-service teachers from elementary education and to access their perceptions about the TBT and the tasks, in order to revise the design principles in case of need. The trails were created in the training institution, based on objects/phenomena inside and outside the school building, and had ten tasks: two concerning visual counting, two concerning combinatorial counting, three on repetition patterns and three on growth patterns, for grades 3–4; and one about visual counting, two about combinatorial counting, two about repetition patterns and five about growth patterns, for grades 5–6. The objects/phenomena were chosen by their mathematical potential but also considering the possibility of their existence in other educational contexts (e.g., windows, mural, trees, bicycle stand, benches, manhole cover, pavements, stairs, lockers, building façade, flag poles, recycling bins). Figures 7–10 illustrate examples of tasks used in the TBT:

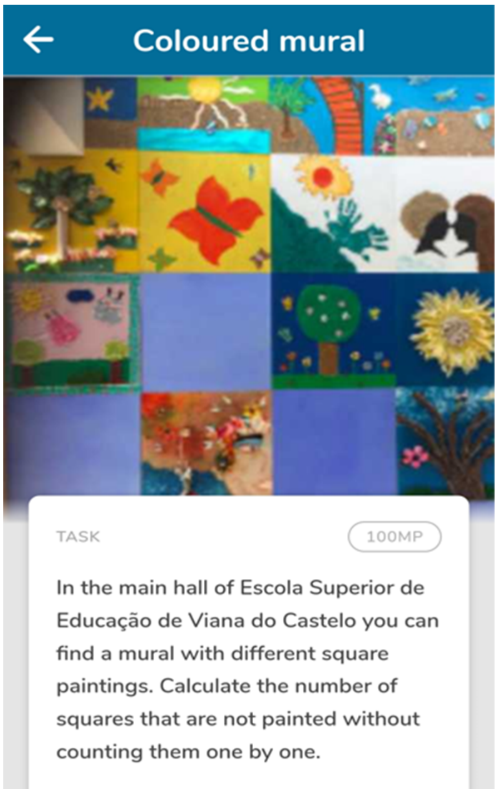

Figure 7. Visual counting task.

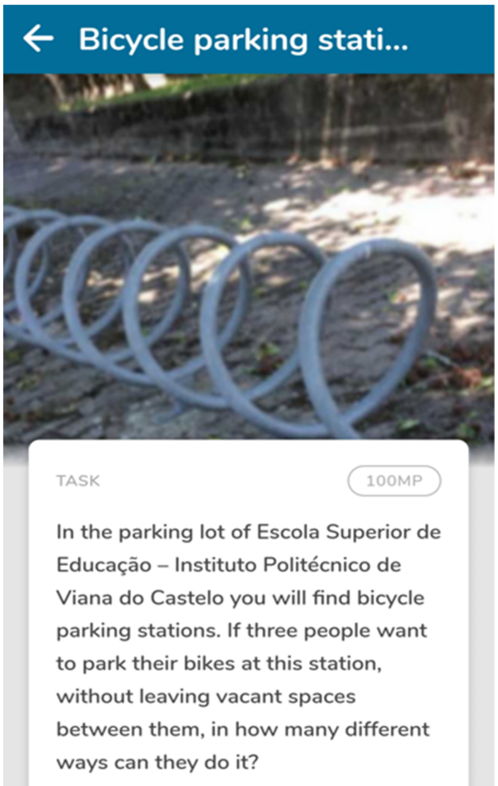

Figure 8. Combinatorial counting task.

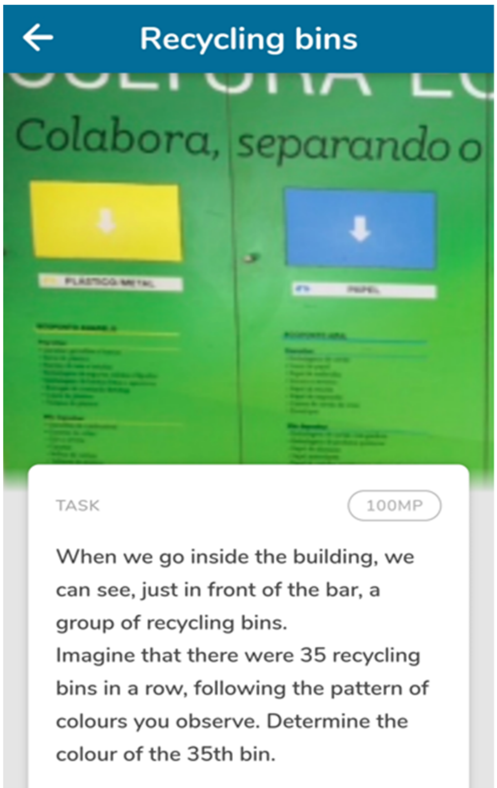

Figure 9. Repetition pattern task.

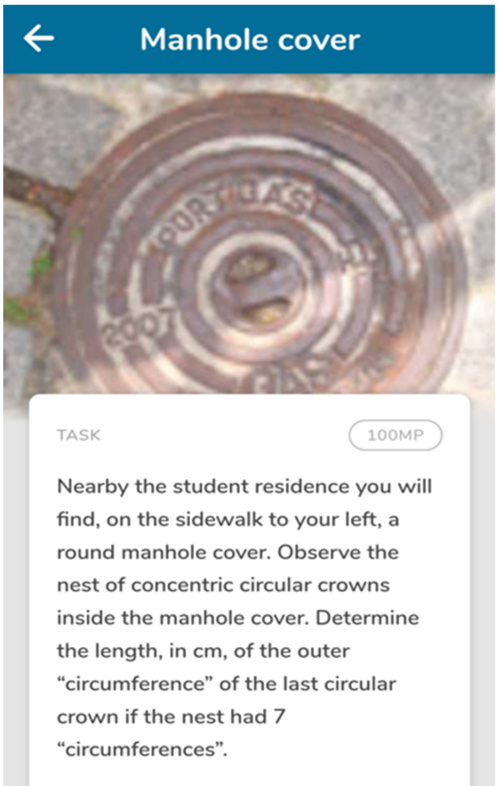

Figure 10. Growth pattern task.

The pre-service teachers experimented on the trails (Figure 11), supervised and observed by the researchers. They already had previous experience with MCM, so there was no need to introduce the app and the mobile math trail dynamic. The focus was mainly on the tasks and the organization of the TBT. After this activity, the participants filled a questionnaire stating their opinions about the TBT, the tasks, MCM use and outdoor education in general.

Figure 11. Pre-service teachers experimenting on the TBT.

The content analysis of the questionnaire, complemented by the field notes, allowed us to conclude that all the participants valued outdoor mathematics education in the form of a trail, stating that: "learning is more meaningful for students"; and "students have a perception that mathematics can not only be taught inside the classroom, but it can also be taught outside, being present in our daily lives and objects that surround us". MCM was recognized as a valuable resource to use, with their future students being considered as a user friendly and intuitive app that promotes autonomy and collaborative work and "gives immediate feedback and allows students to access hints in case of difficulties". MCM was also praised for its contribution to help develop spatial orientation, since the user "has a better perception of the sites where the tasks are placed and the path to follow".

The majority of these pre-service teachers acknowledged the pertinence of a TBT, especially the two TBTs implemented in the scope of Sequences and Patterns, highlighting their usefulness in the application of curricular content, extending the work developed inside the classroom to the outdoors. However, some mentioned that the TBT tasks "would be richer if they also included connections with other mathematical themes", an opinion that is based on experience with tasks (Figure 12) involving other concepts besides the ones of an algebraic nature (e.g., measurement, geometry).

This idea can be considered as a valuable contribution to refine DP3. Following the framework proposed by Smith and Stein [9], good tasks that have the potential to engage students in higher-level thinking induce students to establish connections between procedures and mathematical themes. The participants also commented on the level of cognitive demand of the TBT tasks and were unanimous in agreeing that both types—lower-level and higher-level—should be contemplated.

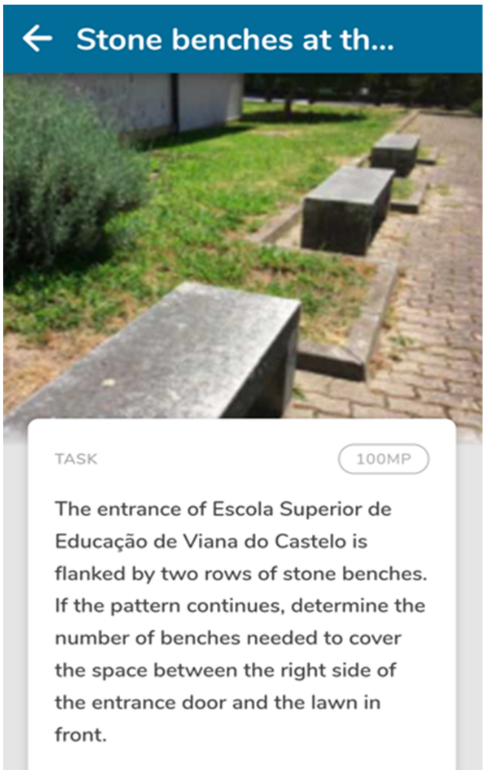

Figure 12. Task of the TBT promoting mathematical connections.

Moreover, concerning the tasks, the participants underlined the importance of clarity in the task definition, defending an objective and concise statement, otherwise it can influence the performance and engagement of the students. Two tasks generated this comment (Figure 13).

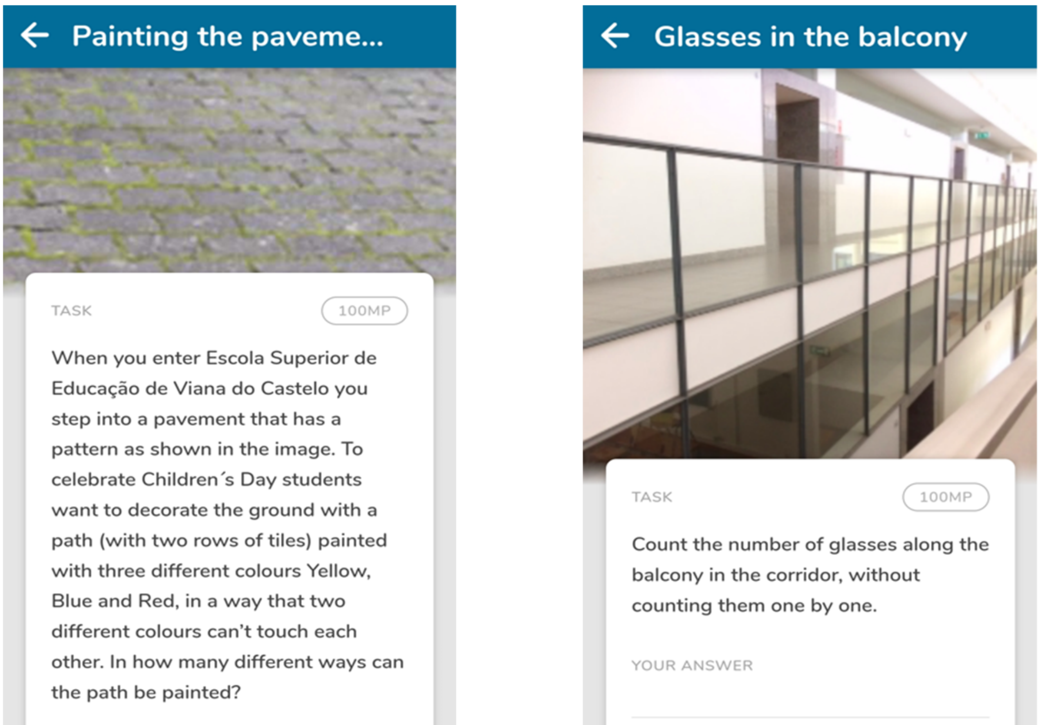

Figure 13. Tasks considered ill-defined by the pre-service teachers.

The pre-service teachers expressed feelings of "frustration" and "disappointment" for not being able to solve the tasks or for reaching an incorrect result. In most of these cases, they mentioned that "it took them a long time to understand the statement" or "the question was not posed in a clear manner" or "was poorly formulated". This is also an important aspect to consider in the design principles regarding task formulation, which influences DP1.

To synthesize, the data analysis resulting from this second cycle produced results that were useful to further develop the design principles for the TBT tasks, introducing changes to DP1 and DP3.

4.3. 3rd Cycle

As expected, the third cycle of this research started by revisiting the design principles, which were informed by the retrospective analysis of the data collected in the previous cycle, introducing changes to DP1 and DP3:

- **DP1—**Formulate a set of particular tasks, in a clear and objective manner, organized in the form of a math trail, based on specific objects/phenomena that can be found in schools, in order to inspire teachers to more easily adapt the proposals to their educational context [46].
- **DP3—**Differentiate the tasks' features in order to diversify the level of cognitive demand [9]. Lower-level and higher-level tasks imply different procedures and reasoning, routine and non-routine approaches, which contribute to a more interconnected mathematical understanding. Balance the level of challenge of the tasks, introducing more lower-level tasks than higher-level ones, and establish connections between the main theme and other mathematical themes.

Taking these principles into consideration, as well as the experience with the design of the TBT used in the second cycle, we developed two TBTs regarding Sequences and Patterns in two different schools, one for grades 3–4 and another one for grades 5–6. Both trails involved the same concepts (visual counting, combinatorial counting, repetition patterns and growth patterns) but had the proper adaptation to the knowledge expected of each age group. In this third cycle, we intended to implement the trails with students from elementary education (4th grade—23 students; 6th grade—25 students) and assess their performance and reactions to the trail as well as their teachers' opinions about the TBT and the tasks, in order to revise the design principles, in case of need. The trails were created in the schools, to avoid organizational/logistic problems, and were based on objects/phenomena inside and outside the school building, and had ten tasks: two on visual counting, two on combinatorial counting, four on repetition patterns and two on growth patterns for grades 3–4; and one on visual counting, three on combinatorial counting, two on repetition patterns and four on growth for grades 5–6. The objects/phenomena were chosen by their mathematical potential, but also considered the possibility of their existence in other educational contexts (e.g., windows, lamps, bicycle stand, benches, pavements, stairs, lockers, building façade, flag poles, recycling bins, hopscotch, chessboard, goal frame, drinking fountain). Figures 14–17 illustrate examples of tasks used in the TBT, representing the main concepts involved. Tasks "Goal Frame" and "Patterns in the water fountain" are representatives of proposals that include connections between the main theme, Sequences and Patterns, and other mathematical topics, like number properties and measurement.

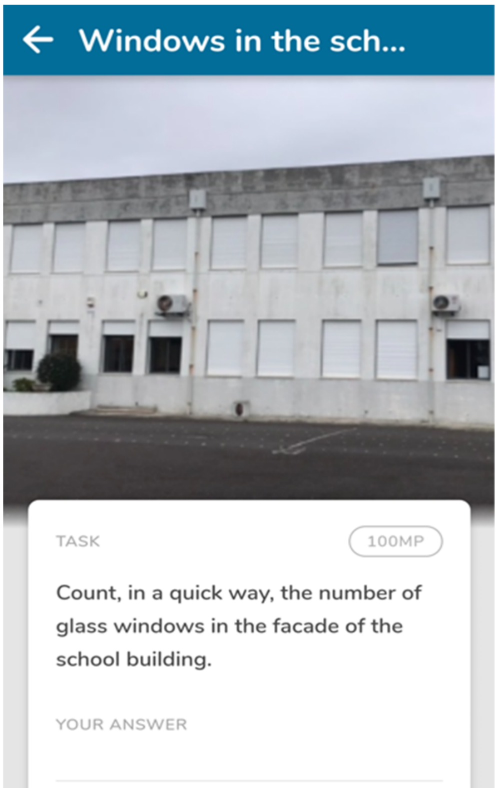

Figure 14. Visual counting task.

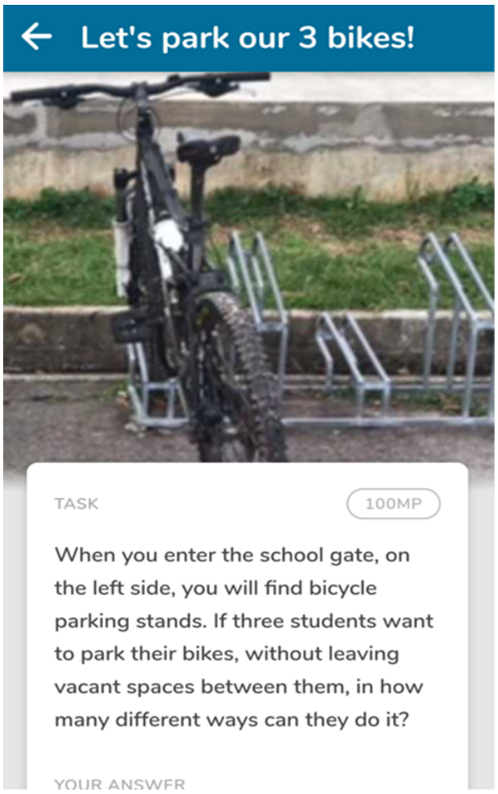

Figure 15. Combinatorial counting task.

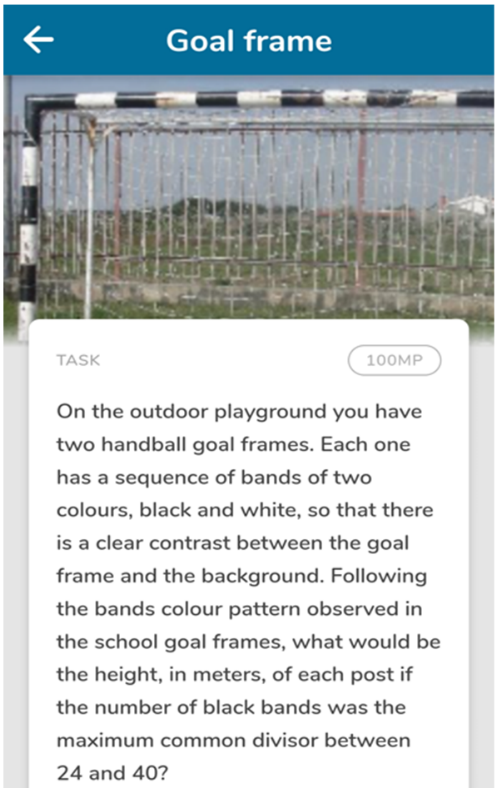

Figure 16. Repetition pattern task.

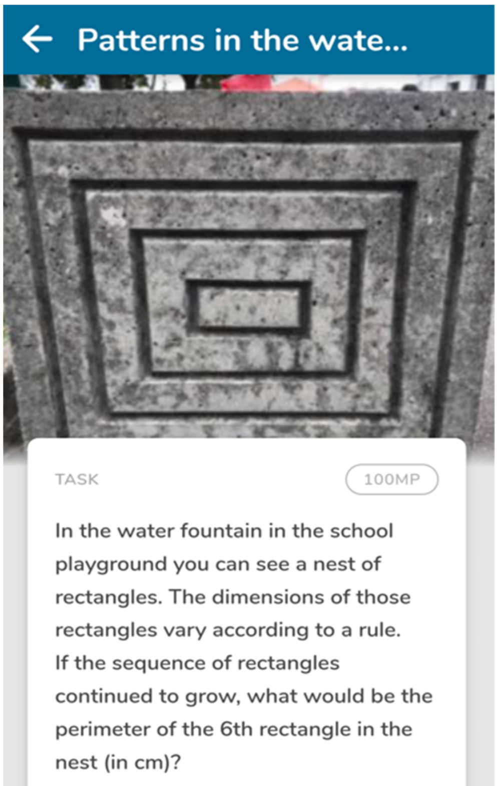

Figure 17. Growth pattern task.

The students in the two schools experimented on the trails (Figures 18 and 19), supervised by their teachers and observed by the researchers. Prior to the implementation students were introduced to the MCM app, getting acquainted with the main features, and the dynamics of the math trail. They worked in groups of 3/4 and had different roles in order to promote collaborative work. One of the elements was responsible for the tablet, reading the tasks and submitting the answers, another one had the tools necessary to collect data (e.g., folding ruler), and the other(s) was in charge of the written records. In general, students did not evidence difficulties with the use of MCM, reacting with excitement to the outdoor activity with technology and showing engagement with the tasks proposed.

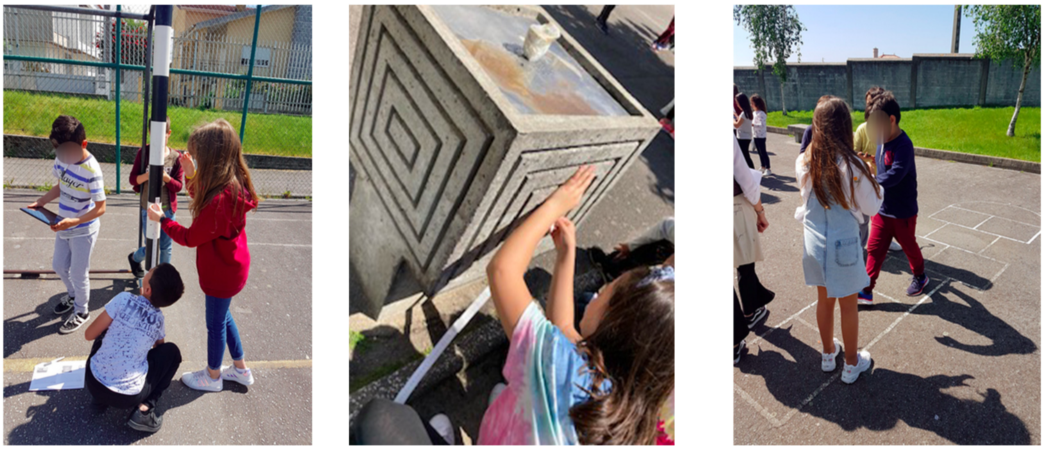

Figure 18. 4th grade students experimenting with the TBT.

Figure 19. 6th grade students experimenting with the TBT.

After the trail implementation, we conducted a semi-structured interview with the two teachers (from the 4th grade and the 6th grade), focusing on aspects like students' reactions to the trail and the tasks, organizational issues relating to the mobile math trail, opinions about the adequacy of the TBT and its tasks. Both agreed on the importance of working in non-formal contexts, like the outdoors, particularly as a complement to the traditional work developed in the classroom. They also highlighted the students' positive reaction to the trail and the use of MCM, being challenged and motivated in the course of the activity, and the efficacy of the collaborative work. In terms of organizational issues, the 4th grade teacher expressed some concerns relating to group management and supervision, which may result in the need for more human resources to accompany the students. However, he also recognized that the option of creating the trail in the school helps overcome this constraint. The two teachers considered that the tasks were globally age-appropriate and adequate for the grade level, "structured according to the curricular guidelines" concerning algebraic thinking. The 6th grade teacher underlined the potential of many of the tasks of the TBT to enable the "simultaneous exploration of other contents, despite being focused on a specific theme" (e.g., geometric figures, similarity of triangles, area, perimeter, properties of quadrilaterals, measurements, visualization, counting, numeric relations, divisors, multiples), which is in line with the ideas already evidenced by the pre-service teachers in the previous cycle.

For these teachers, some of the tasks stood out as being more complex than usual, as they "were not routine tasks", in the sense of being not so common in the work developed in the classroom, as they were "different from the traditional textbooks tasks". Despite this observation, both teachers valued the integration of higher-level tasks in the TBT, as they "provided the opportunity to engage students in higher order reasoning", of a conceptual nature, "which may complement and enrich the work inside the classroom". Regarding the time spent with the trail implementation, both teachers suggested 90 min as being the most adequate. In this sense, they considered that a trail with 10 tasks is inadequate, especially for younger students (4th graders), since, in this case, they needed 2 h for this activity and some requested help to complete the trail in that time frame. Therefore, these teachers proposed that the TBT should have 7–8 tasks, depending on the level of challenge associated with the tasks. This was of particular interest to refine DP1, specifying, in a particular way, the reasonable number of tasks for the TBT.

Based on the results of the experimentation carried out during the third cycle, most of the design principles were validated, either by the reactions of the students or by the teachers' opinions. The suitable number of tasks for a TBT underlined by the teachers made us revise DP1, adding that information. After the application of the three cycles and the evaluation of the results, we reached the following design principles:

- **DP1**—Formulate 7–8 particular tasks, in a clear and objective manner, organized in the form of a math trail, based on specific objects/phenomena that can be found in schools, to inspire teachers to more easily adapt the proposals to their educational context [46].
- **DP2**—In order to develop algebraic thinking in elementary grades, tasks should contemplate the following concepts: counting in visual contexts (subitizing); combinatorial counting; repetition patterns; and growth patterns. This learning trajectory is sustained by research and can be considered as a possible pathway to work on generalization [28].
- **DP3**—Differentiate the tasks' features in order to diversify the level of cognitive demand [9]. Lower-level and higher-level tasks imply different procedures and reasoning, routine and non-routine approaches, which contribute to a more interconnected mathematical understanding. Balance the level of challenge of the tasks, introducing more lower-level tasks than higher-level ones, and establish connections between the main theme and other mathematical themes.
- **DP4**—Formulate the tasks according to the MCM portal conditions, namely the available answer formats and quality criteria [47].

Ultimately, these DPs guided the development of the TBT concerning Sequences and Patterns for grades 3–4 and grades 5–6 published in the MaSCE[3] Project Website (https://masce.eu/the-project/intellectual-outputs/theme-based-trails/theme-based-trails-statistics/theme-based-trails-sequences-and-patterns/, accessed on 12 November 2021), conceived to inspire teachers to develop their own TBT in their schools.

5. Conclusions

In this paper, we presented the development of design principles for so-called TBT in the outdoor mathematics context. In particular, the focus was on early algebraic thinking. We hereby chose a specific context as a learning environment, namely the outdoor environment, and therefore several cycles of task design were necessary to develop a TBT that suited the needs of the national curricula, pre-service and in-service teachers, as well as students who should use and develop algebraic thinking.

Through the different rounds of interventions and feedback from different target groups, it was possible to design suitable tasks in this particular example of algebraic thinking and to specify design principles (DP1–DP4) for TBT in general. Through its explorative approach, the study does not intend to present these design principles as a strict version in the scope of TBT design. Still, it is an attempt to contribute to support teachers in the promotion of algebraic thinking, teaching and learning using the outdoors and a specific digital tool.

In future research, we will adapt and specify the design principles in the scope of further mathematics topics. Whereas TBT for statistics and rational numbers have already been developed on the basis of the design principles, further topics from Geometry and Probability will be created in the context of MaSCE[3].

In addition, the MCM web portal is constantly being improved (e.g., new answer formats), which broadens the possibilities for task design and will be taken into consideration in the implementation of these TBT topics. Through this process, it is our aim to specify and update the design principles in order to invite teachers to use the potential benefits of outdoor mathematics in the scope of different mathematics topics.

Author Contributions: Conceptualization, A.B., I.V., S.J. and M.L.; methodology, A.B., I.V., S.J. and M.L.; validation, A.B., I.V., S.J. and M.L.; formal analysis, A.B., I.V., S.J. and M.L.; investigation, A.B., I.V., S.J. and M.L.; data curation, A.B., I.V., S.J. and M.L.; writing—original draft preparation, A.B., I.V., S.J. and M.L.; writing—review and editing, A.B., I.V., S.J. and M.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: The study was conducted according to the guidelines of the Declaration of Helsinki, and approved by Institutional Review Board of the Polythecnic Institute of Viana do Castelo (protocol code #PP-IPVC-02-2021, date of approval 9 September 2021).

Informed Consent Statement: Informed consent was obtained from all subjects involved in the study.

Data Availability Statement: The data presented in the study are available on request from the corresponding author. The data are not publicly available due to privacy issues.

Acknowledgments: This work was developed in the context of the project MaSCE[3]—Math Trails in School, Curriculum and Educational Environments of Europe (2019-1-DE03-KA201-060118), partially funded by the Erasmus + grant program of the European Union.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. NCTM. *Principles to Actions: Ensuring Mathematical Success for All*; NCTM: Reston, VA, USA, 2014.
2. Vale, I.; Barbosa, A. Photography: A Resource to Capture Outdoor Math. In *Research on Outdoor STEM Education in the digiTal Age. Proceedings of the ROSETA Online Conference, Online, 16–19 June 2020*; Ludwig, M., Jablonski, S., Caldeira, A., Moura, A., Eds.; WTM: Münster, Germany, 2020; pp. 179–186.
3. Kenderov, P.; Rejali, A.; Bartolini Bussi, M.; Pandelieva, V.; Richter, K.; Maschietto, M.; Kadijevich, D.; Taylor, P. Challenges Beyond the Classroom—Sources and Organizational Issues. In *Challenging Mathematics In and Beyond the Classroom—New ICMI Study Series 12*; Barbeau, E., Taylor, P., Eds.; Springer: Cham, Switzerland, 2009; pp. 53–96.
4. Zender, J. *Mathtrails in der Sekundarstufe I. Der Einsatz von MathCityMap bei Zylinderproblemen in der Neunten Klasse*; WTM: Münster, Germany, 2019.
5. Ludwig, M.; Jablonski, S. Step by Step – Simplifying and Mathematizing the Real World with MathCityMap. *Quadrante* **2021**, *30*, 242–268.
6. Vale, I.; Barbosa, A.; Pimentel, T. Math trails a rich context for problem posing—An experience with pre-service teachers. *Quad. Di Ric. Didat. (Mathematics)* **2015**, *25*, 221–227.
7. Gurjanow, I.; Ludwig, M.; Zender, J. Why do in-service teachers and student teachers use MathCityMap and why don't—A short survey on acceptance and user behavior of MathCityMap. In Proceedings of the CERME, Dublin, Ireland, 1–5 February 2017.
8. Doyle, W. Work in Mathematics Classes: The Context of Students' Thinking During Instruction. *Educ. Psychol.* **1988**, *23*, 167–180. [CrossRef]
9. Smith, M.S.; Stein, M.K. Selecting and Creating Mathematical Tasks: From Research to Practice. *Math. Teach. Middle Sch.* **1998**, *3*, 344–350. [CrossRef]
10. Stein, M.K.; Smith, M.S. Mathematical Tasks as a Framework for Reflection. *Math. Teach. Middle Sch.* **1998**, *3*, 268–275. [CrossRef]

11. Vale, I.; Barbosa, A. Visualization: A pathway to mathematical challenging tasks. In *Mathematical Challenges for All*; Zazkis, R., Leikin, R., Eds.; Springer International Publishing: Cham, Switzerland, 2022; Volume 10, in press.
12. NCTM. *Principles and Standards for School Mathematics*; NCTM: Reston, VA, USA, 2000.
13. Barbeau, E.J.; Taylor, P.J. *Challenging Mathematics In and Beyond the Classroom—New ICMI Study Series 12*; Springer: New York, NY, USA, 2009.
14. Hartmann, L.M.; Schukajlow, S. Interest and emotions while solving real-world problem inside and outside the classroom. In *Mathematical Modelling Education in East and West. International Perspectives on the Teaching and Learning of Mathematical Modelling*; Leung, F.K.S., Stillman, G., Kaiser, G., Wong, K.L., Eds.; Springer: Cham, Switzerland, 2021; pp. 153–164.
15. Mullis, I.; Martin, M. *TIMSS 2019 Assessment Frameworks*; TIMSS & PIRLS International Study Center: Chestnut Hill, MA, USA, 2017.
16. Ferreira, P.; Buriasco, R. Math Tasks Instructions Based on the Realistic Mathematics Education Perspective. *Bolema* **2015**, *29*, 452–472. [CrossRef]
17. Borasi, R. On the nature of problems. *Educ. Stud. Math.* **1986**, *17*, 125–141. [CrossRef]
18. van den Heuvel-Panhuizen, M. The role of contexts in assessment problems in mathematics. *Learn. Math.* **2005**, *25*, 2–9.
19. Silver, E. Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM* **1997**, *3*, 75–80. [CrossRef]
20. Stoyanova, E. Problem posing in mathematics classrooms. In *Research in Mathematics Education: A Contemporary Perspective*; McIntosh, A., Ellerton, N., Eds.; MASTEC: Perth, Australia, 1998; pp. 164–185.
21. Baumanns, L.; Rott, B. The process of problem posing: Development of a descriptive phase model of problem posing. *Educ. Stud. Math.* **2021**, *110*, 251–269. [CrossRef]
22. Brown, S.; Walter, M. *The Art of Problem Posing*; Erlbaum: Mahwah, NJ, USA, 2005.
23. Vale, I.; Barbosa, A. Outdoor photography: A resource teacher training. In *Mathematics Education in the Digital Age (MEDA)—Proceedings*; Donevska-Todorova, A., Faggiano, E., Trgalova, J., Lavicza, Z., Weinhandl, R., Clark-Wilson, A., Weigand, H.G., Eds.; Johannes Kepler University: Linz, Austria, 2020; pp. 327–334.
24. Sullivan, P.; Liburn, P. *Good Questions for Math Teaching*; Math Solutions Publications: Sausalito, CA, USA, 2002.
25. Bragg, L.; Nicol, C. Seeing mathematics through a new lens: Using photos in the mathematics classroom. *Aust. Math. Teach.* **2011**, *67*, 3–9.
26. Richardson, K. Designing math trails for the elementary school. *Teach. Child. Math.* **2004**, *11*, 8–14. [CrossRef]
27. Devlin, K. *Mathematics: The Science of Patterns*; Scientific American Library: New York, NY, USA, 2003.
28. Vale, I.; Pimentel, T.; Barbosa, A.; Borralho, A.; Barbosa, E.; Cabrita, I.; Fonseca, L. *Padrões em Matemática—Uma Proposta Didática no Âmbito do novo Programa Para o Ensino Básico. [Mathematical Patterns—A didactical Propose in the Scope of the Elementary Education Curriculum]*; Texto Editores: Lisboa, Portugal, 2011.
29. Barbosa, A.; Vale, I. Visualization in pattern generalization. *J. Eur. Teach. Educ. Netw.* **2015**, *10*, 57–70.
30. Blanton, M.; Kaput, J. Functional thinking as a route into algebra in the elementary grades. In *Early algebraization. A Global Dialogue from Multiple Perspectives*; Cai, J., Knuth, E., Eds.; Springer: Berlin/Heidelberg, Germany, 2011; pp. 5–23.
31. Kieran, C. Overall commentary on early algebraization: Perspectives for research and teaching. In *Early algebraization. A Global Dialogue from Multiple Perspectives*; Cai, J., Knuth, E., Eds.; Springer: Berlin/Heidelberg, Germany, 2011; pp. 557–577.
32. Warren, E. Generalising the pattern rule for visual growth patterns: Actions that support 8 year olds' thinking. *Educ. Stud. Math.* **2008**, *67*, 171–185. [CrossRef]
33. Blanton, M.; Kaput, J. Characterizing a classroom practice that promotes algebraic reasoning. *J. Res. Math. Educ.* **2005**, *36*, 412–446.
34. Rivera, F. *Toward a Visually-Oriented School Mathematics Curriculum: Research, Theory, Practice, and Issues*; Springer: Dordrecht, The Netherlands, 2011.
35. Blanton, M. *Algebra and the Elementary Classroom: Transforming Thinking, Transforming Practice*; Heinemann: Portsmouth, NH, USA, 2008.
36. Arcavi, A. The role of visual representations in the learning of mathematics. *Educ. Stud. Math.* **2003**, *52*, 215–241. [CrossRef]
37. Ball, D.L. Prospective elementary and secondary teachers' understanding of division. *J. Res. Math. Educ.* **1990**, *21*, 132–144. [CrossRef]
38. Vale, I.; Pimentel, T.; Barbosa, A. The power of seeing in problem solving and creativity: An issue under discussion. In *Broadening the Scope of Research on Mathematical Problem Solving: A Focus on Technology, Creativity and Affect*; Amado, N., Carreira, S., Jones, K., Eds.; Springer: Cham, Switzerland, 2018; pp. 243–272.
39. Rivera, F.; Becker, J. Middle school children's cognitive perceptions of constructive and deconstructive generalizations involving linear figural patterns. *ZDM—Int. J. Math. Educ.* **2008**, *40*, 65–82.
40. Shoaf, M.M.; Pollak, H.O.; Schneider, J. *Math Trails*; COMAP Incorporated: Bedford, MA, USA, 2004.
41. Ludwig, M.; Jesberg, J.; Weiß, D. MathCityMap—eine faszinierende Belebung der Idee mathematischer Wanderpfade. *Praxis der Mathematik* **2013**, *53*, 14–19.
42. Cross, R. Developing Math Trails. *Math. Teach.* **1997**, *158*, 38–39.
43. Barbosa, A.; Vale, I. Math trails: Meaningful mathematics outside the classroom with preservice teachers. *J. Eur. Teach. Educ. Netw.* **2016**, *12*, 49–63.

44. Blane, D.C.; Clarke, D. *A Mathematics Trail Around the City of Melbourne*; Monash University, Monash Mathematics Education Centre: Monash, Australia, 1984.
45. Buchholtz, N. Mathematische Wanderpfade unter einer didaktischen Perspektive. *Math. Didact.* **2020**, *43*, 1–16.
46. Ludwig, M.; Jablonski, S. MathCityMap-Mit mobilen Mathtrails Mathe draußen entdecken. *MNU J.* **2020**, *1/2020*, 29–36.
47. Jablonski, S.; Ludwig, M.; Zender, J. Task Quality vs. Task Quantity. A dialog-based review system to ensure a certain quality of tasks in the MathCityMap web community. In Proceedings of the Fifth ERME Topic Conference (ETC 5) on Mathematics Education in the Digital Age (MEDA), Copenhagen, Denmark, 5–7 September 2018; Weigand, H.G., Clark-Wilson, A., Donevska-Todorova, A., Faggiano, E., Grønbæk, N., Trgalova, J., Eds.; University of Copenhagen: Copenhagen, Denmark, 2018; pp. 115–122.
48. Barlovits, S.; Baumann-Wehner, M.; Ludwig, M. Curricular learning with MathCityMap: Creating theme-based math trails. In *Mathematics Education in the Digital Age (MEDA)—Proceedings*; Donevska-Todorova, A., Faggiano, E., Trgalova, J., Lavicza, Z., Weinhandl, R., Clark-Wilson, A., Weigand, H.G., Eds.; Johannes Kepler University: Linz, Austria, 2020; pp. 143–150.
49. Erickson, F. Qualitative Methods in Research on Teaching. In *Handbook of Research on Teaching*; Wittrockk, M., Ed.; MacMillan: New York, NY, USA, 1986; pp. 119–161.
50. van der Akker, J.; Gravemeijer, K.; McKenney, S.; Nieveen, N. *Educational Design Research*; Routledge: London, UK, 2006.
51. Prediger, S.; Gravemeijer, K.; Confrey, J. Design research with a focus on learning processes: An overview on achievements and challenges. *ZDM* **2015**, *47*, 877–891. [CrossRef]
52. McKenney, S.; Reeves, T. *Conducting Educational Design Research: What It Is, How We Do It, and Why*; Routledge: London, UK, 2012.
53. Cobb, P.; Jackson, K.; Dunlap, C. Design research: An analysis and critique. In *Handbook of International Research in Mathematics Education*; English, L.D., Kirshner, D., Eds.; Routledge: London, UK, 2016; pp. 481–503.