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Paper folding for an active learning of mathematics: an experience with preservice teachers

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Abstract. Students are active individuals that build, modify and integrate ideas, interacting with the physical world, materials and other students. Furthermore, not all students learn in the same way, some learn better by seeing, others by listening, others by moving. So, we developed an exploratory study of qualitative nature, with elementary preservice teachers, where the emphasis was on a teaching approach based on problem solving, mathematical communication and collaborative work that involved the manipulation of a sheet of paper. We intended to identify the relevance of this approach with preservice teachers and what were the main difficulties revealed, related to the proposed tasks. Preliminary results allowed to identify several difficulties in terms of mathematical communication and the manipulation of the sheet of paper to solve a problem. However, they reacted positively to the tasks, expressing interest, motivation and recognition of its importance in mathematical learning at any level.

Resume. Les étudiants sont des individus actifs, qui construisent, modifient et intègrent des idées, en interaction avec le monde physique, des matériaux et d’autres étudiants. En outre, tous n'apprennent pas de la même manière: certains apprennent mieux en regardant, d'autres en écoutant, d'autres en bougeant. Nous avons donc développé une étude exploratoire de nature qualitative, avec des futures instituteurs, promouvant une approche pédagogique basée sur la résolution de problèmes, la communication mathématique et le travail collaboratif impliquant la manipulation d’une feuille de papier. Nous voulions déterminer la pertinence de cette approche pour les futures enseignants et identifier les les principales difficultés révélées face aux tâches proposées. Les résultats préliminaires ont permis d'identifier plusieurs difficultés relevant de la communication mathématique et de la manipulation de la feuille de papier pour résoudre un problème. Cependant, les participants ont réagi positivement aux tâches, exprimant leur intérêt et motivation et reconnaissant leur importance pour l’apprentissage des mathématiques à tous les niveaux.

1. Introduction

School mathematics requires effective teaching that engages students in meaningful learning through individual and collaborative experiences, giving them opportunities to communicate, reason, be creative, think critically, solve problems, make decisions, and make sense of mathematical ideas (NCTM, 2014). One of these ways is to use paper folding mathematical tasks, because it is a simple way of discovering and evidencing relations, as well as facilitating the formalization of those relations, contributing to the understanding of different concepts and problem solving (e.g. Coad, 2006). Within Mathematics arises Geometry that traditionally has been a neglected topic in school mathematics and where the experiences provided to the students are mainly based on memorization and training of formulas, overlooking the development of spatial orientation and visualization skills. It is therefore crucial that (future) teachers develop the skills needed to use these materials in the classroom, in a more dynamic and challenging perspective, in which everyone participates actively and collaboratively, communicating with words, but also with gestures, observation and actions.

This communication is part of the M&M - Mathematics&Movement project that assumes that learning involves not only intellectual but also physical activity, and that social interactions, reflected in collaborative work and in collective discussions, are essential components of active learning, facilitating sharing, the development of mathematical meanings and the construction of knowledge. In particular, hands-on tasks are considered to provide learning where students are engaged and become good mathematics thinkers (e.g. NCTM, 2014; Prince, 2004). The aim of this study was to identify and understand the performance and reactions of elementary future teachers when solving tasks through paper folding, identifying the main difficulties.

2. Active learning strategies in mathematics class

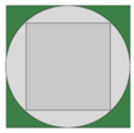
Active learning is usually defined as an instructional method that involves learners in the learning process (Prince, 2004). It requires them to develop meaningful activities and think about what they are doing. Organizations such as NCTM have long been promoting methodologies that require the intellectual engagement in building new knowledge (e.g. NCTM, 2014). However, intellectual engagement may not be enough, in addition to the strategies of intellectual nature. In the context of active learning, those that are arise from social and physical activities are also important. The social interactions of students that mediate learning, must be emphasized in the Mathematics class. This type of collaboration facilitates sharing and development of mathematical meanings. Students, especially the younger ones, need to be physically active in the classroom. In this scope, the use of manipulative materials, the construction of models or the realization of more practical projects, among others are included. Active learning emphasizes the importance of these three strands, noting that learning emerges from experiences and interactions between intellectual, social and physical dimensions (e.g. Nesin, 2012) (Figure 1).

Figure 1. Dimensions of active learning (adapted from Nesin, 2012)

3. Types of thinking: the case of visual contexts

Students often show preferences regarding the way they communicate and how they receive and synthesize information, which presupposes, on the part of the teacher, the consideration of a diversity of types of thinking in the classroom. According to Krutetskii (1976) there are three types of students according to their thinking when they solve mathematical problems: (1) Analytic, those who prefer the use of non-visual resolution methods, choosing logical-verbal methods involving algebraic, numeric and verbal representations; (2) Visual (or geometric), those who prefer the use of visual solution methods, opting for visual-pictorial schemes, involving graphic representation (figures, diagrams, images); and (c) Harmonic (or integrated), those that have no specific preference for logical-verbal or visual-pictorial representations. Reflecting, in particular on visualization, it is relatively consensual that it is fundamental and has great potential, in the sense that it contributes to a global and intuitive perspective for the understanding of different mathematical subjects (e.g. Presmeg, 2014). The following example may serve as an illustration of the importance of visualization and of visual skills.

A circle is inscribed in a square, then another smaller square is inscribed in the circle. Find out the area of the smallest square knowing that the area of the largest square is 100 cm^2?



(an equivalent question is to ask *what is the ratio of the area between the two squares*).

The traditional way to solve it is to use the theorem of Pythagoras and formulas of the area of the circle and of the square. However, if we see the smaller square in another positon, the solution is immediate and free of errors of calculation or unknowledgeable formulas. Figure 2 shows a visual sequence of steps to achieve the solution (1/2 x 100 = 50).

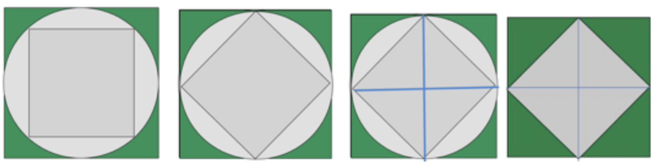


Figure 2. Visual sequence of the steps

This is a simple, clear and elegant solution, or a dynamic solution (Kruteskii, 1976; Presmeg, 2014). We learn to think visually or to "see to think", and therefore we can also teach it, which is important (e.g. Presmeg, 2014; Whiteley, 2004).

Sometimes it is easier to perceive or even explain a concept by creating an image, since it is quickly understood and retained longer, than a sequence of words. So, paper folding can help all students, mainly the non-visual ones.

4. Paper folding: an active and visual approach

Manipulative activities provide learning experiences that promote curiosity and interest, facilitating the introduction of formal concepts through the use of concrete materials. The actions of folding applied to the paper allow it to be transformed into different forms, either two-dimensional or three-dimensional, opening the opportunity to investigate and discover relationships of different nature. In this way, paper folding can be a dynamic, creative and appealing strategy to approach several concepts in the mathematics class, facilitating visualization and problem solving (Coad, 2006).

Levenson (1995) mention the effective contribution to the development of mathematical thinking as well as the ability to communicate mathematically with proficiency and group interaction skills. Sze (2005) recognizes the potential of paper folding in the refinement of mathematical vocabulary, in particular associated with geometry. Also in the scope of communication, the construction of precise models with aesthetic sense from folding leads to the students having to listen and observe with extreme attention all the instructions verbalized/presented (e.g. Levenson, 1995; Sze, 2005). In addition to the natural relationship of paper folding with mathematics, which facilitates its integration into the math classes, close relationships with the types of thinking can also be identified (Boakes, 2009). The work of paper folding involves students physically, requiring auditory abilities and visual stimuli, and it is through these actions, which involve spatial skills, that mathematical ideas and meanings are constructed and discussed.

5. Methodology

As previously mentioned this report is part of a larger study with pre-service teachers. In this teaching experience we followed 45 students, future teachers of elementary education (3-12 years old), that attended a unit course of Didactics of Mathematics, where a sequence of tasks was proposed in which folding and cutting paper were used as an active learning strategy. The tasks allowed to highlight transversal skills, particularly communication, and problem solving within the scope of Geometry, and were presented in visual contexts. We adopted a qualitative methodology of exploratory nature, collecting data, in a holistic, descriptive and interpretative way, through observation and written productions, regarding the proposed tasks, and written comments on the experience. After gathering the data, it was sought to find patterns related to the main difficulties identified and expressed by the future teachers when solving the proposed.

6. The tasks and some preliminary results

For this paper, we selected two paper folding tasks: the *heart*, highlighting mathematical communication, and the *cube*, highlighting visual problem solving.

6.1. Task 1: The heart

The heart: With a paper strip one student gives instructions to another colleague, positioned on his/her back, to produce a heart through paper folding

This task aims to show the importance of correct communication of instructions that may lead to a particular folding, in which the emitter does not see what the receiver does. The teacher slowly folds a rectangular paper to get a heart that the emitter students observe (half of the students). The teacher performed the demonstration without any oral instructions, so it was up to the students' emitters to decide what actions to communicate to their receiver pair, who did not see the construction (Figure 3).

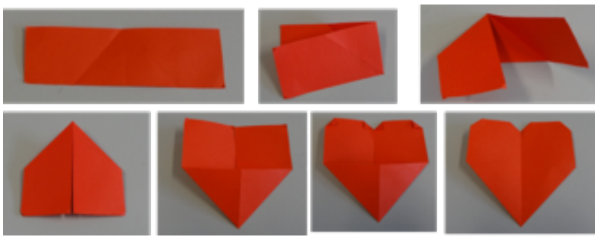


Figure 3. Folding of the heart performed by the teacher

To do this, each emitter would be on his/her back with his/her colleague receiver, which would be instructed to do the folding, starting with a rectangular paper as well. With this dynamic, neither the emitter student nor the receiving student had access to what the other element was doing. By observing the work of the peers, some students used references to geometric terms, others showed more difficulties and did not use adequate mathematical language and/or clear information, which conditioned the result. In some cases, the receiver did not interpret the message the way the sender had idealized by folding the paper in the wrong way in at least one of the steps. Most of the students felt the need to use gestures along their description, even knowing that they were not being observed by their pairs. Figure 4 shows the dynamic of the students during the paper folding experience and some of the final results of the heart, demonstrating that some of them didn't obtain the expected folding.

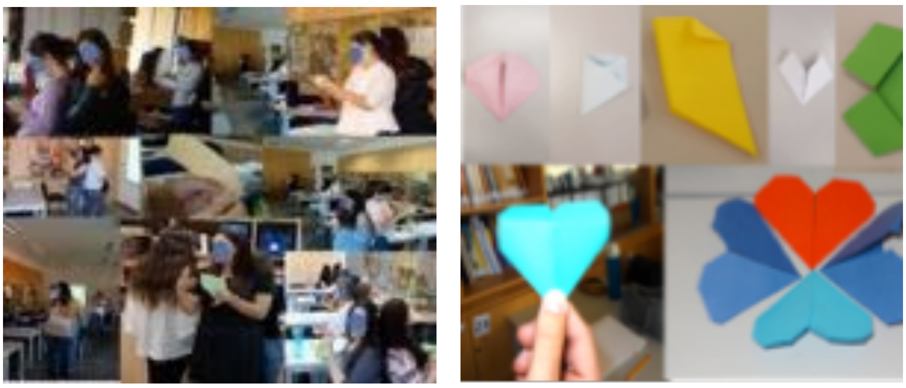


Figure 4. Dynamic and some results of the paper folding task

6.2. Task 2: The cube

The cube: Use a square sheet of paper to draw the net of a cube with the maximum volume. Then construct the cube by folding that net.

This problem involves geometric and spatial reasoning. Many nets can be built on a square sheet, but only one fits the conditions. It is a problem with some complexity for the elementary level. The students mostly began by exploring the most obvious possibilities in which the segments representing the edges in the planning are parallel to the sides of the square sheet of paper or take advantage of the diagonal of the square.

Despite different trials to reach a solution, none of them led to the expected solution (Figure 5).

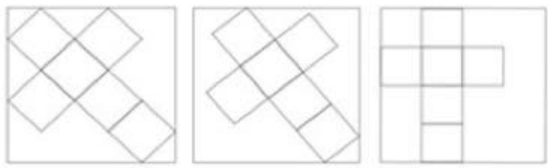


Figure 5. Incorrect solutions presented by the students

After discussing the correct solution, they discovered that their proposals did not have the highest volume. Figure 6 shows one of the analytical productions where they compare the volume with two of the nets.

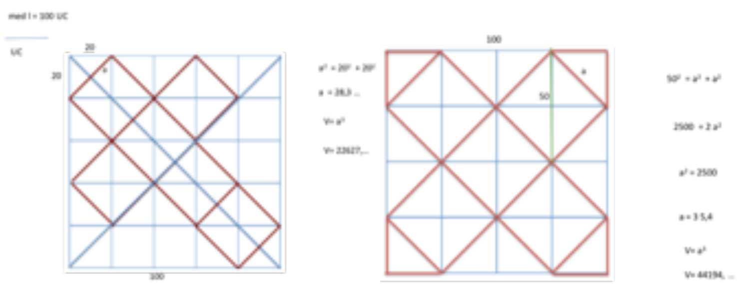


Figure 6. Analytical solutions to determine the volume

The option of the students was to make the design of the possible traditional nets. In fact, it is necessary to have mathematical knowledge to apply to the situation, and also intuition linked to the visualization of the different nets of a cube. In addition, exploration required divergent thinking in order to imagine and admit a completely different net from the classical approaches. Another way to approach it was the use of trial and error, doing the folds on the square paper and coming up with more positive results. After discovering the wright net, they did many attempts to get the cube, but not all of them got the solution by themselves (Figure 7).

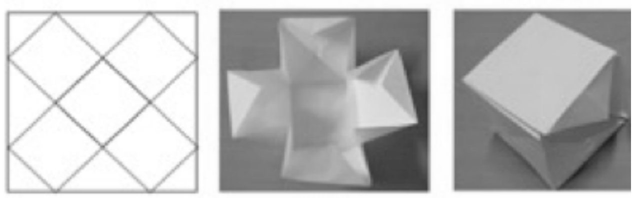


Figure 7. Correct solution and paper folding cube

This was a problem with some complexity, for the elementary level, but students were engaged and the discussions that emerged at the end allowed a better understanding of the importance of the use of different approaches to solve a mathematical situation.

7. Some concluding remarks

We can identify several difficulties in terms of mathematical communication and the manipulation of a sheet of paper to solve a problem. Students were not familiarized with paper folding tasks especially in mathematics classes. We can conclude that the majority of these students were not visual, possibly due to the lack of previous experiences during their academic course. They had interpretation difficulties related to visualization, with geometric and spatial concepts, as well as difficulties in using correct language, recurring to imprecise and/or unclear terms.
Despite the difficulties, students recognized the potential of the tasks and reacted positively, expressing interest and engagement. As students, they assumed that these tasks could improve mathematical communication. As future teachers, they concluded that paper folding is a more dynamic and direct way of assessing and improving students' mathematical communication and improving the learning of mathematical concepts.

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