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Mathematical problems: the advantages of visual strategies

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Abstract
The rapid evolution of today's world requires that all students have access to an education that values creativity, critical thinking and problem solving. It means motivating students to use multiple strategies when solving a problem, including the visual ones as an important support in solving all kinds of problems, including those in which the visual component is not evident. So, teachers should include practices that lead students to think visually and develop this ability through experiences that require such way of thinking. In this context, we discuss the advantages of using visual strategies when solving problems with multiple approaches, illustrating those ideas with some examples.

Key words: Problem solving, problem solving strategies, visualization, creativity, thinking styles.

Introduction

The 21st century school must prepare students for a global society oriented by high-speed communications, with great visual impact, and by complex, diversified and rapid changes at all levels. Many of the current jobs require that the employees think outside of the box so they can solve situations from different angles, and thus be able to build a new way of thinking. Being in a highly competitive society, we face several problems in our schools, one of them is having students that are strongly unmotivated to learn mathematics. We must be able to inspire them to want to learn, to continue to learn, to understand what they are learning, to learn more and to enjoy learning mathematics. In this context, the role of the teacher is quite demanding (Vale, 2017). Among other aspects, it is the teachers' responsibility to select the most appropriate educational resources, in particular, the tasks to be proposed, since different tasks, with different levels of cognitive demand induce different ways of learning (e.g. Stein & Smith, 1998). Thus, teachers should include in their practices diversified tasks that go beyond routine situations, focusing particularly on problem solving, which constitutes an essential theme in mathematics teaching and learning with extreme importance in the curriculum. However, students' problem-solving skills still require substantial improvement. One challenge that we have to grasp in our classroom practices is reversing this situation by helping students construct their mathematical knowledge with problem solving. According the National Council of Teachers of Mathematics [NCTM] (2014) to ensure that students have the opportunity to engage in higher order thinking, teachers should regularly select and implement tasks that promote reasoning and problem solving.

We can identify a number of problems, usually of a visual nature, that allow the use of diversified approaches, however, it is known that in the mathematics classroom visual strategies or visual solutions are rarely used and/or valued (e.g. Barbosa & Vale, 2015;

Stein & Smith, 1998). Recent research in the area of cognition (e.g. Boaler, Chen, Williams, & Cordero, 2016)., in particular about problem solving processes, indicates that for certain types of tasks, the use of strategies that requires visual representations may have advantages over the use of other representations, facilitating problem solving. This fact should be faced with concern if we consider the type of teaching that is promoted, since visualization is not only a powerful tool in approaching geometric topics, but also in others (e.g. numerical field), therefore teachers should be prepared and motivated to use it in their classroom practices.

Taking these ideas into account, and based on studies carried out in the context of preservice teacher education (3-12 years old), this paper discusses the potential of visual strategies/solutions in/of some mathematical problems. After a brief theoretical background, we highlight advantages of visualization, used as a powerful strategy in solving problems, through the analysis of some examples.

Problem solving – visual strategies

Before discussing the meaning and usefulness of visual solutions in problem solving, we start by synthetizing some ideas about the importance of exploring problems with multiple solutions and the teaching of problem solving strategies as a contribution for divergent thinking and, consequently, for a more effective mathematical understanding.

Traditionally, a problem is considered as a situation that involves the student in significant mathematical activity, but for which he does not know an immediate path to reach the solution. So, facing a problem, it is necessary to choose and use methods and strategies that must be thought and adequate for each situation. Problem solving strategies are very powerful when attacking a problem because they are tools that are often identified with reasoning processes and can be very useful in the different steps associated to the problem solving process. So, it is relevant to assume that mathematical knowledge and reasoning strategies must be learned and used simultaneously and not in isolation (e.g. making a drawing, making an organized list, finding a pattern, trial and error, working from the end to the beginning) (Vale & Pimentel, 2004).

Although the visual has been underestimated for several decades, at the end of the 20th century, there was a resurgence of interest in spatial reasoning and visualization as powerful tools in mathematical reasoning, since it may help students to go beyond the mere use of formulas, stimulating the development of intuition and the ability to see new relations, producing the cut with mental fixations contributing to a broader view of mathematics. Visualization has often been cited as a process in mathematical learning, recognized as a component of reasoning, deeply involved with conceptual knowledge rather than just perceptual, problem-solving and proof, and can be an important aid to all types of problems, including those in which the visual component is not evident (e.g. Arcavi, 2003; Presmeg, 2014; Rivera, 2011; Zimmerman & Cunningham, 1991). According to Zimmermann and Cunningham (1991), visualization is the process of forming images (mentally, using paper and pencil or supported by technology) and using those images effectively in mathematical discovery and understanding, whereas Clements and Battista (1992) define spatial reasoning as the set of cognitive processes by which the mental representations of objects, relations and spatial transformations are constructed and manipulated.

Both spatial reasoning and visualization play vital roles not only in geometry and geometry education but also in mathematics and mathematics education (e.g. Giaquinto, 2007; Presmeg, 1986, 2014; Rivera, 2011). Since in the literature various terms are used, such as visualization, visualizing, spatial thinking, spatial reasoning, spatial visualization, visuospatial thinking and visual reasoning, we will use the term visuospatial reasoning to emphasize the spatial, visualizing (imagistic and as representations that others can see), and reasoning aspects, elements related to the solvers' visual skills (e.g. Sinclair, Bussi, Villiers, Jones, Kortenkamp, Leung & Owens, 2017). In this paper, when we use the term visual we refer to visuospatial reasoning.

According to Fischbein (2002), intuition is "a special type of cognition characterized by self-evidence and immediacy" (p. 200), being generated by experience, that is, by practical situations in which the individual is systematically involved. When solving a problem, intuition is fundamental because it is the moment that gives sense and certainty that the path to a solution is found and, from this moment, the problem is seen in a simpler way, or, as Liljedahl (2004, p.21) points out "intuition provides the direction to follow". Some authors (e.g. Liljedahl, 2004; Presmeg, 2014) argue that solving a problem may come as a sudden idea, a kind of *aha!*, like a flash of insight, which can be understood as a mental process obtained without great effort, as a sudden understanding of something, after a period of trying unsuccessfully to solve a problem, and which arises without a clear understanding of where and why it arises.

During the process of solving a task, visual solutions are considered to include the use of different visual representations (e.g. figures, drawings, diagrams, graphics) as an essential part of the solution process. Non-visual solutions, on the other hand, do not depend on visual representations as an essential part to reach the solution, resorting to others, such as numerical, algebraic and/or verbal representations (e.g. Presmeg, 1986; Vale, Barbosa & Pimentel, 2016). So, visual representations can help students to progress in their understanding of mathematical concepts and procedures, giving meaning to the mathematical contents involved in problems and simultaneously either solving the problem or during the collective discussions, since the drawings and other visual supports are of particular importance for students with difficulties, helping them to follow the reasoning and to share it with their colleagues (e.g. Arcavi, 2003; Kruteskii, 1976; NCTM, 2014; Vale, 2017).

All teachers, throughout their practice, have certainly found students who show preference for some themes, like numbers or geometry, that manifest in the way they understand and solve problems, giving preference to words, formulas or figures. These aspects have to do with the thinking styles of each student, which the teacher must keep in mind, to plan the classroom strategies (Vale, 2017). According to Krutetskii (1976), there are two types of thinking, logical-verbal and visual-pictorial, and it is the balance between these two modes of thinking that determines how mathematical ideas operate in an individual. Thus, mathematical psychologists and mathematical educators (e.g. Krutetskii, 1976; Lowrie & Clements, 2001; Presmeg, 1986, 2014) identify three categories in relation to the problem solving processes, connected with the strategies used by students, that we (Vale, Pimentel & Barbosa, in press) adapted as: (1) visual or geometric – those who prefer to use visual-pictorial schemes even when the problems are more easily solved with analytical means, that is, they prefer a holistic approach to the problem, favoring visual methods to solve a certain problem, which can be solved visually or analytically; (2) non-visual, analytical or verbal – those who prefer to use non-visual, more verbal approaches, even in those problems that are relatively simpler to solve with a visual approach; and (3) mixed, integrated, or harmonic – those who have no specific preference for either logical-verbal or pictorial-visual thinking, and tend to combine analytical and visual methods. Some studies show that students, taught in a visual way, tend to learn to use visual methods in problem solving, and others show that there are students who prefer to use visual strategies while others prefer more verbal or analytical approaches (e.g. Presmeg, 2014). In our experience (e.g. Barbosa & Vale, 2015; Vale et al., in press), we identify visual students, but we also find that this way of thinking is possible to be taught. Students often tend to use algebraic ways to process information over visual ones, even when the former are more complicated - a trend that often leads to disastrous results because these students lack the mathematical knowledge to do a thorough analysis of the problems or, despite having them, they lack the understanding (e.g. Eisenberg & Dreyfus, 1991).

Our vision about problem solving includes teaching that provides students with a range of non-routine problems in which different strategies can be applied. In particular, we value the *seeing* strategy, which may be complementary to other strategies, since visualization may be a powerful alternative approach that increases the window of possibilities for problem solving, providing solutions different from the most traditional ones. So, we are giving students the opportunity to be creative, what means to use fluency, flexibility and originality. This strategy, of thought, involves the visual perception of mathematical objects combined with past knowledge and experience. In addition, *seeing* is related to having creative insights or *aha!* moments, it can be expressed in terms of drawing and involves an activity that can be associated with the more traditional range of strategies (e.g. make an organized list, look for a pattern), often generating a more understandable, simple, and creative visual resolution of a problem (Vale, Pimentel & Barbosa, in press).

Some potentialities of visual solutions

As Krutetskii (1976) points out, one of the characteristics of mathematically competent students is being able to look for a clear, simple, short, and therefore elegant solution to a problem. Research increasingly emphasizes the need for good visuospatial students, however teachers must organize a set of tasks to help them in their practice in order to show the advantages of this work. The potentialities and limitations of visuospatial reasoning is recognized as part of the classroom mathematical culture (e.g. Arcavi, 2003; Presmeg, 2014, Sinclair et al, 2017). Thus, in order to develop students' visuospatial skills, as well on preservice and prospective teachers, we present some examples that admit visual and non-visual solutions, that can be used from the most elementary levels, and with which we intend to highlight the potential of visual solutions. According to Presmeg (2014), these examples can be classified as visual problems, due to their context, or presentation, or because the visual solution is more powerful. For this paper we opted for examples of geometric nature.

No one can explain exactly what goes on in our heads when suddenly we have an idea that leads us to the solution or to the path that leads us to the solution. This moment of insight, or *aha!* experience, is most often associated with problem solving in visual contexts. This is described as being a creative moment (Lilhedahl, 2005). Of all possible solution strategies some are clearly simpler and involve an *aha!* experience (e.g. Liljedahl, 2004; Presmeg, 2014; Vale et al., in press), leading to more original solutions. Some problems may be complicated for those individuals who are analytical in the solutions they adopt, or, at least can be more laborious due to the number of calculations they require. However, after the discovery of the visual relations involved, with some intuition or *aha!* experience to begin the solving process, these problems become much simpler and more evident, hence accessible to more students. Let's analyze the example in Figure 1 (Vale, 2017).

Example 1

The figure represents a square whose midpoints form another square. The midpoints of the new square form another square.
What is the ratio between the area of the largest square and the area of the shaded square? Discover more than one process of solution.

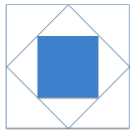


Figure 1. Task 1

The most usual solution makes use of the Pythagorean theorem to determine the length of the sides of the two smaller squares, starting from the length of the side of the initial unit square, to obtain the desired area. The simplest solution to this problem depends on finding the relation between the different parts in which the square is divided, which only happens from the moment the student tries to see possible relations between the several figures. This can occur through intuition. In this case it is necessary to draw the two diagonals of the larger square (Figure 2).

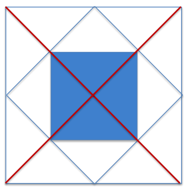


Figure 2. Visual solution for task 1

At this point, using the properties of the different squares, the initial square is divided into sixteen equivalent (and geometrically equal) parts, and the answer easily comes up. The requested ratio is 4/16 or 1/4. Another similar visual solution could be the one presented on Figure 3. Instead of identifying the diagonals of the larger square, we can start with the diagonals of the middle square, seeing that we obtain a square mesh covering the whole figure, then we follow the previous conditions.

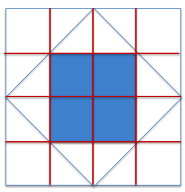


Figure 3. Alternative visual solution for task 1

The creativity of the solution manifests itself in breaking with the established set of knowledge (e.g. Haylock, 1997; Presmeg, 2014), such as the use of formulas and the routine type of solution suggested by the word area, and also in choosing more visual processes that show that the inner square has a quarter of the area of the original, as suggested in Figures 2 and 3. In this example, the use of the *seeing* strategy is not essential because more traditional analytical strategies could have been used. However, this strategy, by allowing an alternative solution, simplifies the process of solving the problem and, at the same time, allows us to relate other knowledge and develop the flexibility that underlies divergent thinking, which is one of the characteristics of creativity.

Example 2

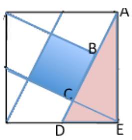
Another potentiality of the *seeing* strategy emerges before problems in visual contexts, where the figures are presented or result of the translation of the problem, allowing the discovery of certain properties of the figures that favors the justification of numerical conclusions, avoiding calculations or the use of formulas.
Consider the following task (Vale, Pimentel & Barbosa, 2015) (Figure 4):

Draw a square and connect each vertex with the midpoint of the opposite side, as shown in the figure.
What is the area of the blue square?

Figure 4. Task 2

This problem can motivate several solutions involving the properties of the observed figures. Students who attempt a solution using formulas applied to different parts of the figure may find this problem difficult to solve. The usual solution for this type of problems, once the area is requested, is to look for numerical relationships between the initial square and the blue square, privileging an analytical solution. Two of the most common analytical solutions are presented in Figure 5.

Solution 1:
Determine the length of the side of the square:



$\Delta[AED]$ and $\Delta[DCE]$ are similar

$$\frac{\overline{AD}}{\overline{DE}} = \frac{\overline{AE}}{\overline{CE}} \qquad \frac{\frac{\sqrt{5}}{2}}{\frac{1}{2}} = \frac{1}{\overline{CE}} \qquad \overline{CE} = \frac{1}{\sqrt{5}}$$

$$\frac{\overline{CD}}{\overline{DE}} = \frac{\overline{CE}}{\overline{AE}} \qquad \frac{\overline{CD}}{\frac{1}{2}} = \frac{\frac{1}{\sqrt{5}}}{1} \qquad \overline{CD} = \frac{1}{2\sqrt{5}}$$

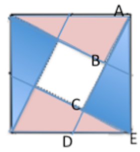
$$\overline{CB} = \frac{\sqrt{5}}{2} - \left(\frac{1}{\sqrt{5}} + \frac{1}{2\sqrt{5}}\right) = \frac{1}{\sqrt{5}}$$

$$\therefore A_q = \left(\frac{1}{\sqrt{5}}\right)^2 = \frac{1}{5}$$

Using the Pythagorean Theorem

$$\overline{AD}^2 = \overline{AE}^2 + \overline{DE}^2$$
$$\overline{AD}^2 = 1^2 + \left(\frac{1}{2}\right)^2$$
$$\overline{AD} = \frac{\sqrt{5}}{2}$$

Solution 2:
Determine the area of each of the four equal right-angled triangles:



Using the Pythagorean Theorem

$$\overline{AD}^2 = \overline{AE}^2 + \overline{DE}^2$$
$$\overline{AD}^2 = 1^2 + \left(\frac{1}{2}\right)^2$$
$$\overline{AD} = \frac{\sqrt{5}}{2}$$

$\Delta[AED]$ and $\Delta[DCE]$ are similar

$$\frac{\overline{CD}}{\overline{DE}} = \frac{\overline{CE}}{\overline{AE}} \qquad \frac{\overline{CD}}{\frac{1}{2}} = \frac{\frac{1}{\sqrt{5}}}{1} \qquad \overline{CD} = \frac{1}{2\sqrt{5}}$$

The area of each triangle

$$\overline{AC} = \overline{AD} - \overline{CD} = \frac{\sqrt{5}}{2} - \frac{1}{2\sqrt{5}} = \frac{2}{\sqrt{5}}$$

$$At = \frac{\overline{CE} \times \overline{AC}}{2} = \frac{\frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{5}}}{2} = \frac{1}{5}$$

$$Aq = 1 - \left(4 \times \frac{1}{5}\right) = \frac{1}{5}$$

Figure 5. Analytical solutions for task 2

In these solutions, only calculations are used, there is no attempt to *see* a relation between the different parts of the figure. The simplest solution for this problem appears when the students focus on *seeing* possible relations, which can happen through intuition. If they are able to *see* those relations, they may discover a dynamic visual solution, mentally rotating the triangles on the inside of the square to the outside of the square. The figure obtained shows that the area of the blue square is 1/5 of the area of the initial square (Figure 6).

Figure 6. Dynamic visual solution for task 2

It is possible to find alternative visual solutions. In Figure 7 we may observe another example. Dividing the initial square into 20 equal triangles, we conclude that $\frac{A_q}{A_Q} = \frac{4}{20} = \frac{1}{5}$

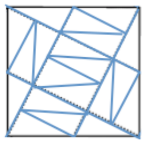


Figure 7. Other visual solutions for task 2

It is important to highlight that solutions like these can be considered as creative, by the fact of breaking with some mental schemes like the use of formulas, suggested by the word *area*, presenting a simpler and more significant method.

Example 3

> Students go to school using different means of transportation. One third of the students go by bus. One quarter of the remaining students goes by car and the others walk or take a bike to school. Knowing that 90 students go to school by car, how many students attend this school?

Figure 8. Task 3

In this example, the whole is an unknown quantity, which normally makes this problem complex for students. The most common numerical solution is: start by determining the part of the whole that remains when you take away the part of the students that use the bus, that is, $1 - \frac{1}{3} = \frac{2}{3}$. Calculating the part of the students that go by car, $\left(\frac{1}{4}\right)$ of this quantity $\left(\frac{2}{3}\right)$, we have $\frac{1}{4} \times \frac{2}{3} = \frac{2}{12} = \frac{1}{6}$. As the part of the students that go by car is $\frac{1}{4}$, or 90 students (the sixth part of the whole), the whole will be 6x90 = 540. We conclude that the school has 540 students. This solution involves not only a procedural knowledge but also a conceptual knowledge that not all students can mobilize simultaneously, according to the conditions of the problem.

It is also possible to recur to a visual solution in this type of problems. Using a visual model, students can begin by representing the unknown quantity (number of students) with a bar. Then, they will try to see the given information in the successive bars.

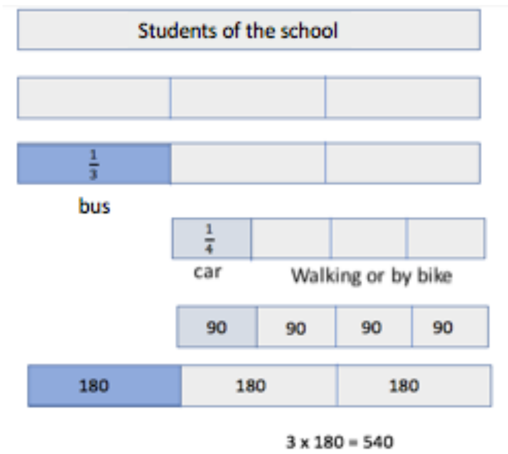


Figure 9. Visual solution for task 3

In alternative we can use only one bar, concentrating all the needed information:

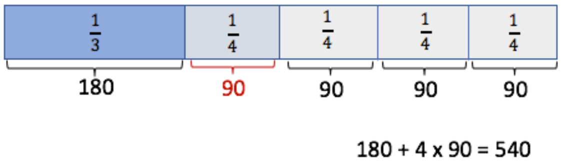


Figure 10. Another visual solution for task 3

In an early phase of the learning process, a visual resolution is easier for students to understand than an analytical solution. It is important for the teacher to work simultaneously analytic and visual solutions, showing the correspondence between the two, in order to give meaning to the mathematical expressions and the words involved. Subsequently, the students will choose the best strategy.

Example 4

What fraction of this rectangle is painted in red?

Figure 11. Task 4

This is a typical problem that involves a geometric progression, associated to a visual context. The most common solution, and the option chosen by all the students, implies the discovery of the number of red parts that the rectangle is divided in, which means finding the value of the expression: $S = \frac{1}{8} + \frac{1}{8} \times \frac{1}{8} + \frac{1}{8} \times \frac{1}{8} \times \frac{1}{8} + \cdots$, that is, $S = \frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3} + \cdots$ Being a geometric progression with ratio $\frac{1}{8}$ and first term $\frac{1}{8}$, then we will have $S = \frac{\frac{1}{8}}{1-\frac{1}{8}} = \frac{1}{8-1} = \frac{1}{7}$, the fraction that represents the part painted in red. It is important to state that the students were not successful in solving this task, using this

particular approach, either by not remembering the formula associated to a geometric progression or by presenting mistakes in calculus.
Although this was the privileged approach, a visual solution was possible by analysing the pattern underlying the division of the rectangle. The initial figure is divided in 8 squares, each one representing $\frac{1}{8}$ of the total, where 7 are painted in different colours. The last square is divided in 8 rectangles, each one representing $\frac{1}{8}$ of $\frac{1}{8}$ of the initial rectangle. This happens recurrently. Each colour covers exactly the same area of the initial figure, no matter the number of subdivisions. As there are 7 colours, each one represents $\frac{1}{7}$ of the rectangle. This solution is useful when students do not have the necessary tools to perform an algebraic solution. Furthermore, this approach could complement the analytical solution, clarifying the meaning of the algebraic expressions.

Throughout this section we intended to emphasize the importance of visualization in solving certain mathematical problems, illustrating with examples some of its potentialities. However, we are aware that the visual approach is not extensible to all types of problems and may not always help. For example, when an image is given in a problem it tends to influence the student's way of thinking. However, sometimes the representations used (e.g. figures, diagrams) may contain too much information that may be difficult to interpret or have scarce information that does not allow for a complete and correct visual reasoning or even be presented with lack of rigor. The possible potentialities and limitations of a visual aid will depend on the type of information that the student will be able to remove (or add) to the given image, which depends on his/her previous knowledge and the past teaching experiences (e.g. Arcavi, 2003; Presmeg, 2006). Thus, the teacher should value visualization using different types of visual representations (e.g. drawings, gestures, movements) and motivate students to use them, but also present different examples of the same concept, avoiding prototypic examples. The teacher must be aware of these aspects in order to adjust practices to develop students' visual skills, as a further contribution to mathematical understanding (Vale, Pimentel & Barbosa, in press).

Concluding Remarks

The way in which problems are usually posed suggests the use of non-visual, analytical (e.g. numeric, algebraic) solutions, so students are expected to use them, but it is important, no matter how efficient a solution of this nature is, to show that there are other equally valid ways to solve the same problem. Analytic solutions, though being powerful and general, may be opaque to some students, even if they do not realize it. Many are able to successfully manipulate symbols and procedures when solving a problem and are satisfied even if they do not understand very well what they have done. However, as Arcavi (2003) points out, they can never acquire and develop the meaning of the symbol. For these situations, other alternatives should be sought. Visual strategies may be one of these alternatives because they have the potential to allow students to understand that the blind manipulation of symbols and procedures is not always possible and can be the necessary and complementary level for more formal and complex comprehension.

This paper highlights the strength of the *seeing* strategy in visual solutions. The presented examples show that this approach can be used in different topics of mathematics, not only in geometry. We advocate the use of problems with multiple solutions, in order to ask students for more than one way to solve it, expecting that a visual solution emerges, allowing the development of flexibility of thinking, and consequently contributing for the development of students´creativity. Among the possible strategies to solve a problem, some are clearly simpler than others, involving an *aha!* moment (e.g. Liljedahl, 2004; Presmeg, 2014) leading to more original solutions. The thinking style presented by students is usually analytical and in some cases integrated. Thus, the role of the teacher is fundamental: he/she must have a wide range of tasks to be used for this purpose, and when visual strategies do not appear naturally, the teacher should draw attention to this way of thinking to increase students' repertoire of strategies (Vale, Pimentel & Barbosa, in press) and thus contribute to an improvement on problem solving.

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