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Mathematical reasoning: the impact of visual patterns

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Abstract. This presentation is part of a study that sought to understand how 6th grade students solve visual patterning tasks. It explores some results from four case studies to illustrate generalization strategies privileged by students and difficulties emerging from their work. Another goal is to reflect on the role of visualization in their reasoning and on some factors that may influence the choice of generalization strategies in such contexts.

Introduction

Mathematics education is one of the more globalized areas in education, mainly because of its objective nature and its contribution to technological and economic development. Problem solving has been progressively becoming a fundamental part of the teaching and learning process in mathematics and is considered an unavoidable mathematical ability, one that all students must develop (NCTM, 2000). This prerogative is stated in the recent curricular guidelines of several countries, including Portugal. However, some international studies (e.g. TIMSS, PISA) have shown that Portuguese students perform badly when solving problems (Amaro, Cardoso & Reis, 1994; OCDE, 2004). It must be said that the overall achievement of Portuguese students has improved in the last PISA study, in 2009, but we are still in the lower positions of the ranking which, along with the difficulties observed in classroom experiences, constitute a matter of concern to the community of researchers and educators.

This study approaches problem solving through the exploration of visual patterning tasks: (1) pattern generalization may contribute to the development of abilities related to problem solving, emphasising the analysis of particular cases, organizing data in a systematic way, conjecturing and generalizing, promoting at the same time the emergence of algebraic reasoning; (2) work with patterns may be helpful in building a more positive and meaningful image of mathematics (NCTM, 2000); (3) visualization is an essential mathematical ability, with potential to facilitate learning in areas beyond Geometry, but its role has not always been emphasized in students’ mathematical experiences (Presmeg, 2006). Visual reasoning can help students make sense of concepts and mathematical ideas, allowing them to overcome some of the difficulties associated to the reckless use of procedural knowledge.

Aim

This study aims to understand how 6th grade students (11-12 years old) solve problems involving visual patterns, addressing the following research questions: (a) How can we characterize students’ generalization strategies?; (b) Which difficulties do 6th grade students present when solving visual patterning tasks?; (c) What is the role played by visualization on students’ reasoning?

From patterns to generalization

Generalization plays a crucial role in the activity of any mathematician, being considered an inherent ability to mathematical thinking in general. In the case of the curricular context, we can also state that it is a key goal in the learning of mathematics. It is a means of communication, a tool of thought, which is the basis for the development of mathematical knowledge and the center of activity in this area. The search for patterns has been associated with generalization, considering that it could lead naturally to the expression of generality (e.g. Mason, Johnston-Wilder & Graham, 2005; Orton & Orton, 1999). Such tasks can be a powerful vehicle for understanding relationships between quantities that underlie mathematical functions, thus contributing to the establishment of relations of functional type (Blanton & Kaput, 2005; Warren, 2008). On the other hand, they constitute a concrete and transparent way for students of lower levels to start debating with the notions of generalization and abstraction. It is expected that, through this approach, students be able to more easily attribute meaning to the language and the symbolism used in algebra and in the corresponding representational systems, such as graphs and tables.

The role of visual patterns in finding functional relations

The importance given to visualization in the learning of mathematics is based on the fact that it is not limited to mere illustration, but also being recognized as a component of reasoning (Vale, 2012). Although not being an easy task, the integration of visual approaches in students’ mathematical experiences is suggested (NCTM, 2000). There are two major challenges: most students associate mathematics to the manipulation of numbers, numeric expressions and algorithms, which can contribute to the depreciation of visualization; and the teacher should take into consideration that there are many ways of *seeing* (Duval, 1998). Visual characteristics can be captured in two ways: perceptually and discursively. Perceptual apprehension occurs when the figures are seen as a whole object, a single unit. In discursive apprehension spatial arrangements that compose the figures are identified, either individually or in relation to each other, as a configuration of objects that are related through an attribute or invariant property.

Tasks that involve the study of patterns can be proposed in various contexts, visual and non-visual, and give rise to different approaches. However, the literature reports that the use of a visual support in problems related to the search of patterns, can lead to the application of different approaches to generalization, either of visual or non-visual nature (e.g. Barbosa, 2010; Stacey 1989; Swafford & Langrall, 2000). Also, visual patterns may generate different rules that enhance: connections between arithmetic and geometric relationships; the attribution of meaning to the rules formulated; the need to formulate and validate conjectures. Thus, working with functional relationships through visual growth patterns may facilitate the attribution of meaning to the operations that transform the independent variable in the dependent variable.

In the context of visual patterns, students that are able to analyze figures, in a discursive way, can do it in different ways. They may identify sets of disjoint elements that are combined to build the initial figure, thus giving rise to a constructive generalization (Rivera & Becker, 2008). On the other hand, they can observe the existence of overlapping configurations, counting some elements more than once that are subsequently subtracted, which means that the generalization formulated is deconstructive in nature (Rivera & Becker, 2008). Several studies have concluded that students tend to use more constructive generalizations than deconstructive generalizations (e.g. Barbosa, 2010; Rivera & Becker, 2008), since the latter category involves a higher cognitive level in terms of visualization.

Generalization strategies in visual contexts

Curricular guidelines of several countries reflect an enthusiastic view about the role of patterns in mathematics, highlighting the transversal nature of this theme. The Portuguese curriculum refers the importance of developing abilities like searching and exploring numeric and geometric patterns, as well as solving problems, looking for regularities, conjecturing and generalizing (ME-DGIDC, 2007). Pattern generalization is achieved by the application of a given strategy, but different students may use different approaches to accomplish generalization. In this sense, there has been significant research about students' generalization strategies. An analysis of frameworks proposed by several investigators (Lannin, 2005; Orton & Orton, 1999; Rivera & Becker, 2008; Stacey, 1989) led me to develop the following categorization (Barbosa, 2010).

***Table 1*. Generalization Strategies Framework**

Strategy		**Description**
Counting (C)		Drawing a figure and counting its' elements.
Whole-object	No adjustment (W_1)	Considering a term of the sequence as unit and using multiples of that unit.
	Numeric adjustment (W_2)	Considering a term of the sequence as unit and using multiples of that unit. A final adjustment is made based on numeric properties.
	Visual adjustment (W_3)	Considering a term of the sequence as unit and using multiples of that unit. A final adjustment is made based on the context of the problem.
Difference	Recursive (D_1)	Extending the sequence using the common difference, building on previous terms.
	Rate - no adjustment (D_2)	Using the common difference as a multiplying factor without proceeding to a final adjustment.
	Rate - adjustment (D_3)	Using the common difference as a multiplying factor and proceeding to an adjustment of the result.
Explicit (E)		Discovering a rule, based on the context of the problem, that allows the immediate calculation of any output value given the correspondent input value.
Guess and check (GC)		Guessing a rule by trying multiple input values to check its' validity.

This categorization identifies several generalization strategies, considering patterns in visual contexts. They are, however, differences detected in their nature, considering that, in some cases, figures play an essential role in discovering the invariant (C, TU_3, D_3, E), and in others the work is developed in a numeric context (TU_1, TU_2, D_1, D_2, GC). We can conclude that different strategies can be used to solve the same problem but, depending on the characteristics of the situations presented, it is essential that students understand the strengths and limitations of each strategy, becoming flexible in their reasoning.

Method

Given the nature of the problem posed, I performed a qualitative approach (Erickson, 1986), following a case study design (Yin, 1989). The study implicated fifty-four 6^{th} students (11-12 years old), from two different classes, over the course of a school year. These students solved seven tasks working in pairs. Two pairs from each school constituted four case studies, being selected for clinical interviews. The tasks applied required *near* and *far generalization*, featuring linear patterns as well as non-linear ones, presented in visual contexts.

The data collected was essentially descriptive in nature, resulting from three fundamental sources: observation, interview and document analysis. All the sessions corresponding to the implementation of each task were observed and videotaped, for later viewing and analysis. After these sessions semi-structured interviews were conducted with the case studies, having been audio recorded and transcribed. The interviews allowed to clarify aspects related to students' difficulties, as well as the types of strategies used by them. The document analysis was based on all the registers produced by the students, on biographical records related to the educational background, and field notes written by the researcher throughout the study.

The data analysis process followed the model proposed by Miles and Huberman (1994). Data resulting from interviews, observation of each session, as well as the registers produced by the students, were reduced in order to identify patterns and classify information for further interpretation. These categories focused particularly on generalization strategies used by students and difficulties identified. In the first instance a categorization was designed based on literature, that was gradually refined through the process of collecting and analyzing data, having emerged new categories.

Results and Discussion

Throughout the tasks a variety of strategies were identified, either of visual and non-visual nature. This fact has to do with the structure of the tasks. With visual patterns students can choose to apply a visual or an analytic approach. They revealed a tendency to apply the counting strategy (Figure 1) in near generalization questions and to formulate a rule (explicit strategy) in far generalization questions (Figure 2).

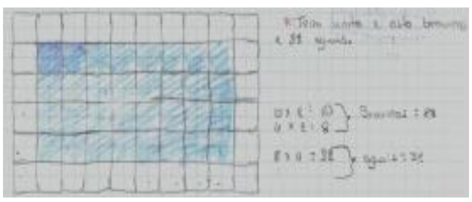

Figure 1. Determine the number of tiles in a 10x6 pool

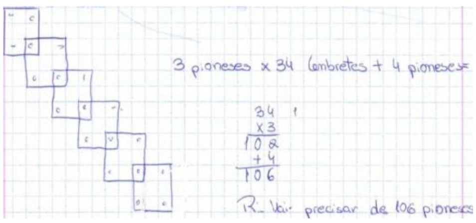

Figure 2. Determine the number of pins necessary for 35 cards

As expected, students achieved better results in near generalization questions than on far generalization questions, having sometimes trouble finding a rule that related the variables. It was also noticed that reverse thinking was complex for many of them, even with some experience with patterning activities, acquired along the study. In these cases, they found an alternative to the explicit strategy and many privileged the guess and check strategy (Figure 3), to overcome the difficulty of using the inverse operations, having found previously a general rule.

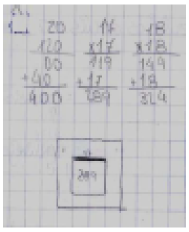

Figure 3. Find the dimensions of a pool, having at your disposal 300 blue tiles

Some of the pairs worked exclusively on number contexts using inadequate strategies like the application of direct proportion, using multiples of the difference between consecutive terms without a final adjustment and mixing variables. Along the study, this tendency was gradually inverted as most students understood the limitations of some of those strategies. It's important to state also that the numbers used on the task may also influence the strategy choice by students. Attractive numbers, in a multiplicative sense, may lead students to recur to a proportional reasoning (Figure 4), even when it's not adequate.

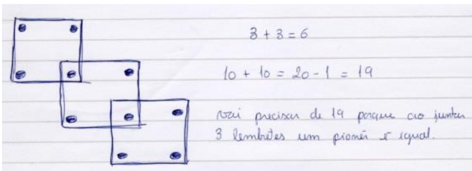

Figure 4. Number of pins needed to hang 6 cards

In some cases, students revealed difficulties finding a functional relation, frequently generalizing rules that verified for particular cases or showing a fixation for a recursive strategy. The conception they had of proving and the fact that their previous experiences were mainly of numeric nature may explain these facts.

The figures representing the pattern may be *transparent* or *non-transparent* in nature, if the structure of the pattern is readily identifiable in the figure or not, respectively. In the case of non-transparent figures students had greater difficulty in deducing a rule, presenting a tendency for recursive numerical approaches (Figure 5). Rarely got to be successful in the attempt to identify a rule directly from the figures (Figure 6), since their apprehension was compromised (Duval, 1998).

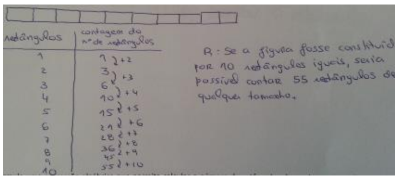

Figure 5. Number of rectangles of different sizes in a sequence of 10 unit rectangles

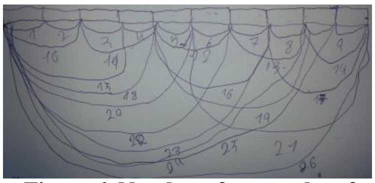

Figure 6. Number of rectangles of different sizes in a sequence of 10 unit rectangles

Visualization proved to be a useful ability in different situations, like making a drawing and counting its elements, in near generalization, and "seeing" the structure of the pattern, finding an *explicit* strategy. The application of visual strategies allowed students to find different equivalent expressions to represent the same pattern, which promoted divergent thinking, and gave them the opportunity to understand the meaning of each expression. It is important to provide tasks which allow the application of a diversity of strategies and encourage students to use and understand the potential of visual strategies, establishing a relationship between the numeric and visual contexts to better understand the meaning of numbers and variables. The connection between parallel approaches and the exploration of the potentialities and limitations of each strategy can contribute to the development of a more flexible reasoning, essential to problem solving.

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