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Mathematics and art: Ideas to use in teacher education

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Abstract

Throughout history, there have always been connections between mathematics and various forms of art, such as music, dance, painting, sculpture, and architecture, which can be effectively used in mathematics classes with highly positive implications for students' learning. In particular, geometry emerges as a central field in this relationship. These connections not only motivate students and promote mathematical learning but also foster creative thinking and an esthetic sense, both of which are intrinsic to many mathematical concepts. Furthermore, they enable students to recognize and understand the mathematics embedded in numerous visual artistic works. While art provides an esthetic experience for those who study mathematics, it also establishes a connection between thought and emotion, giving deeper meaning to what we perceive. However, this relationship is not always immediately visible. Therefore, both mathematics and art teachers need to use various connections to develop both the student's "geometric eye" and "artistic eye". This manuscript aims to discuss ideas about this connection between mathematics/geometry and art. In addition, it presents examples of tasks implemented during the initial training of elementary school teachers, along with some of the works produced by these future teachers.

Keywords: Connections; Elementary teacher education; Visual arts; Mathematics education; Geometry

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1. Introduction

There is a mutual benefit for both mathematics and art when we explore interesting connections between the two fields. This begins with understanding that mathematics does not just involve formulas but also encompasses patterns, symmetry, structure, form, numbers, and beauty – concepts that are accessible to everyone. Mathematics generates and inspires art, and connecting the two allows for a more intuitive, engaging, and playful approach to mathematics. The perception of school mathematics is often that it is difficult, uninteresting, and serves primarily as a filter for selecting individuals wishing to pursue further studies. This negative image of mathematics must be challenged in schools. Mathematics is accepted when it is understood, and one effective way to achieve this understanding is through art. Students should be exposed to the full potential, beauty, power, and usefulness of mathematics. By teaching it as a living science relevant to the individual across various fields of societal domains, students can become educated, active, critical, and adaptable members of society. On the other hand, mathematics has played a pivotal role in shaping modern culture, as well as a vital element of that culture, as Morris Kline pointed out back in 1953. The mathematician stated that it is undeniable that mathematics has shaped the course of modern history, even if sometimes imperceptibly. It is more than just a method, an art, or a language; it constitutes a body of knowledge that is useful to scientists of different specialties – from physicists to artists – and shapes diverse fields, from politics to theology. In addition, mathematics satisfies the curiosity of those who scrutinize the heavens, reflect on the harmony of musical sounds, or ponder the simplicity of daily actions.[1]

In this manuscript, we will discuss some of the relationships between mathematics and art, as recommended in the European educational guidelines,[2] through selected mathematical concepts and artistic productions. These connections can have a positive impact on learning, by fostering creative thinking and an esthetic sense of mathematical concepts while also helping students recognize the mathematics embedded in many artistic expressions. This text focuses on some mathematical concepts found in artistic productions related to visual arts, including painting and architecture. This choice is justified from a curricular perspective, where connections between mathematics and other subjects are advocated.[3] We will also present examples of tasks used in elementary school mathematics, during initial teacher training. These examples illustrate the connections between mathematics, art, and real life.[2-5]

2. The intersection of mathematics and art

Since ancient times, mathematics has been an essential tool for artistic creation, both with concepts and tools – from the compass to the most sophisticated dynamic geometry programs. Until the Renaissance, the separation between art and mathematics made little sense, as exemplified by the universal genius, Leonardo de Vinci. Throughout history, mathematics, particularly geometry, and art, have been linked, developing and complementing each other as interconnected fields of knowledge.[2] Mathematical concepts such as golden numbers, symmetry, spatial orientation, proportions, geometric elements, projections, and fractals demonstrate the vital role mathematics has played in artistic productions. There are, therefore, numerous human productions in which this link has been established, illustrating the intentional and often intuitive and naïve use of mathematical concepts by artists and craftsmen in the search for esthetic balance when creating their works.[2]

Qualities such as creativity, beauty, universality, symmetry, dynamism, and precision are frequently attributed to both art and mathematics. Mathematics has a remarkable potential to reveal structures and patterns, enabling us to better understand the world around us. It fosters the ability to generate ideas, create, and dream! It allows us to imagine different worlds, and it also gives us the chance to communicate these ideas clearly and unambiguously. This unique ability to enrich the imagination in a structured way has long attracted artists. History has shown that mathematics has often evolved for esthetic reasons.[1]

Aristotle argued that philosophers who dismiss the connection between mathematics and esthetics are mistaken, as beauty is a fundamental focus of mathematical reasoning and demonstrations. For instance, Hardy[6] translated this idea, stating that the mathematician, like the painter or the poet, is a creator of patterns. A painter makes patterns with shapes and colors, a poet with words, and a mathematician with ideas. All patterns are inherently beautiful. Ideas, like colors, words, or sounds, must fit together perfectly and harmoniously. In this way, Hardy and other mathematicians describe mathematics as a form of art or a creative activity, with its beauty arising both from their work and from mathematics itself. There is, therefore, an indisputable relationship between art and mathematics. While the two domains seem to depend on different patterns of thought, their structures are strikingly similar. There is no question about the relationship between science and mathematics, where the scientific process is clearly subordinate to mathematics. But the link with art, although seemingly less obvious, has existed since time immemorial – not only with mathematics but, more recently, with technology. This connection, particularly between mathematics and technology, is exemplified through Mandelbrot's fractals, which are discussed later in this article.[7]

Mathematics and art share many common attributes, and teaching and learning both can significantly benefit students. Engaging art in its broader and inclusive sense, across its different dimensions (e.g., visual art, drawing, painting, architecture, music, dance, poetry), offers a more intuitive and playful way to approach mathematics. This process also highlights the esthetic dimension of the mathematical edifice as a whole, i.e., as the science of patterns – a facet that is not traditionally emphasized in teaching. Unfortunately, mathematics is often presented to students in ways that focus on its least engaging aspects. To address this, students should be given the chance to see mathematics as more than just a collection of techniques, algorithms, formulas, and rigid, indecipherable concepts.[5]

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Art, mathematics, and creativity can form powerful partnerships in the classroom, benefiting both students and teachers. Students appreciate opportunities to create a poem, formulate a problem, design a foldable, craft a geometric composition, or build an eye-catching box of chocolates in response to assigned challenges. The results of these efforts are both positive and rewarding. Teachers, in turn, face the challenge of addressing students with different abilities and varying ways of processing information. These divergent learning processes can significantly enhance students' curiosity and performance, meeting their different learning styles and interests. The diversity of tasks and teaching strategies, which reinforce connections and incorporate various forms of languages, can contribute to the proficiency we strive to achieve.

3. An encounter between mathematics and art: A few flashes

The works of various artists, such as Klee, Kandinsky, Vasarely, Escher, and Corbusier, showcase the profound influence of mathematics. These artists drew inspiration from mathematical concepts, exploring new optical possibilities, creating algorithms, and geometries (e.g., non-Euclidean and fractal), further enhanced by advancements in computing, sound, and other technologies.[8] Among the countless relationships that can be established, explored, and analyzed in various artistic productions, mathematics has played a vital role in artistic creation from antiquity to the present day. Certain mathematical concepts, which hold significant relevance in art, can be effectively incorporated into teaching through artistic contexts. These include geometric elements, the notion of pattern, the golden rectangle, proportion, symmetry, and fractals.

The question "What is mathematics?" has inspired extensive discussion over the years, as the concept is dynamic and continuously evolves. The understanding of mathematics in ancient times differed from that of the Renaissance, in the last century, and will certainly continue to evolve in the future. However, a definition attributed to Sawyer,[9] which has gained widespread acceptance among mathematicians and educators due to its simplicity and timeless relevance, states: "Mathematics is the science of patterns," a definition with which we are comfortable. This idea aligns with Steen's[10] assertion that "What mathematicians do best is discover and reveal hidden patterns". Hardy[6] compares the mathematician to the painter or poet, stating that "The mathematician, like the painter or the poet, is a master of patterns," thus harmoniously establishing connections between ideas.

One of the most emblematic works by the Renaissance master Albrecht Dürer (1471 – 1528) is the engraving Melancholia I (1514) (Figure 1). This work integrates numerous mathematical elements and has been the subject of extensive academic discussion. It prominently features a polyhedron, specifically a truncated rhombohedron, which has sparked debates and is often referred to as the "Dürer Solid".[11] In addition, the engraving includes a numerical figure, a magic square, whose formation obeys a pattern, presenting a challenge for those who wish to build one. A magic square is a grid divided into smaller squares, denoted as an $n \times n$ square, with an arrangement of integers in rows and columns such that the sum of the numbers in each row, each column, and both diagonals is always the same, known as the magic sum. The magic square in question is a 4×4 grid, with a magic sum of 34. Another numerical detail appears on the last line of this magic square, in the central squares, indicating the year it was created – 1514. This magic square is believed to be the first to appear in Europe after a 250-year gap following its origins in China. This example, like many others, highlights the fascination artists have had with mathematics (e.g., why is the magic square featured in this work?). It also demonstrates their extensive mathematical knowledge, which allowed them to create a harmonious piece.

Many of the works by various artists throughout history, created using simple geometric elements, such as lines, polygons, solids, and angles, can be effectively incorporated into mathematics classes. These works will not only help students develop their "geometric eye" but also foster their creativity and artistic sensibility (Figure 2).

Other, more complex works can be explored by students at higher levels. Some artists often use geometric figures and basic mathematical concepts to create works that depict seemingly impossible constructions. One of the most creative artists in this field was the Dutch artist Maurits Cornelis Escher (1898 – 1972), who can be considered a mathematical artist due to his exceptional esthetic sense and vivid imagination, combined with a profound understanding of mathematics. This allowed him to create pieces that not only conveyed his unique perspective on the world but also challenged our senses. Many of his works exemplify these ideas (Figure 3).

Symmetry was extensively explored by Escher and other artists and can considered one of the most powerful concepts that spans various themes in mathematics. However, we must clarify the term "symmetry", which was used for many years in mathematics curricula until the 2007 Basic Education Mathematics Program in a different sense. Before 2007, confusion arose from the translation of terms from English, such as axial symmetry and central or point symmetry, which should more accurately be referred to as reflections, half-turns (in the plane), and inversions

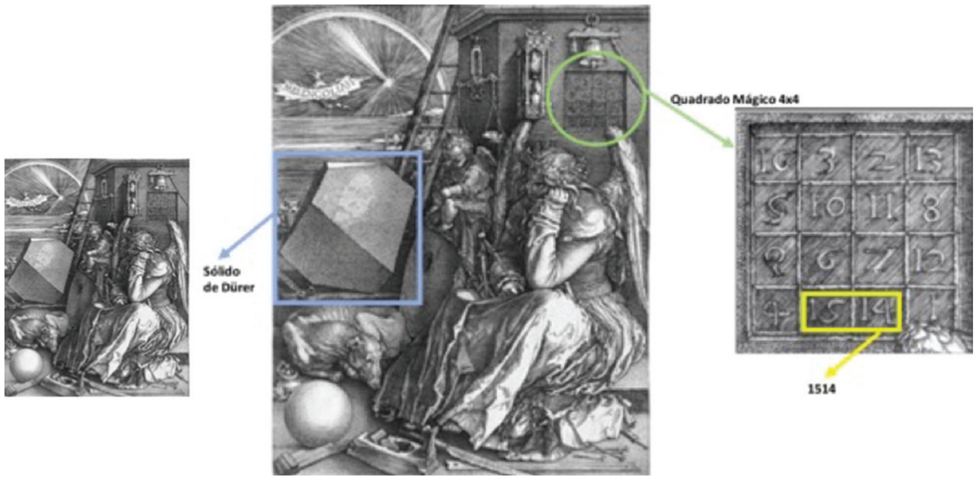


Figure 1. Melancholia I by Albrecht Dürer
Source: Images from Getty Images and the authors.

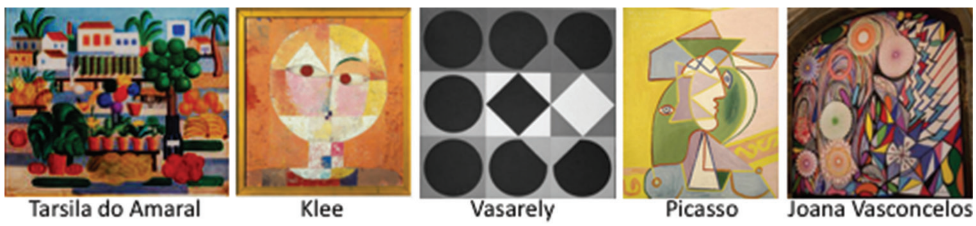


Figure 2. Paintings with basic geometric elements
Source: Images from Getty Images.

Figure 3. A photograph of Escher (A) and some of his masterpieces (B)
Source: Images from Wikipedia (A) and Pixabay (B).

(in the plane). Thus, when we refer to symmetry, we are talking about the symmetry of a figure, which correlates with various geometric transformations.[12,13] In geometry, symmetry is defined in terms of isometries. Specifically, when the image of a figure, through an isometry other than identity, coincides with the original figure, the figure is said to possess symmetry.[14] Symmetry is a fundamental geometric characteristic present in both nature and art. It is found throughout the natural world and is also one of the most prevalent themes in painting, architecture, and design, spanning cultures worldwide and throughout human history. By examining the images in Figure 4 and asking "What do all these images have in common?", the answer is symmetry.

Symmetry is both beautiful and fascinating, which is why students are often captivated by concrete and tangible examples of it in nature and art. The study of symmetry can range from elementary to highly advanced; for example, one can start by identifying the symmetries in designs and patterns or delve into the use of symmetry groups as an accessible way to introduce the abstract framework of modern mathematics. Furthermore, the principles mathematicians apply in the study of symmetry are not exclusive to mathematics and can be found in other areas of human thought. By exploring symmetry in different contexts, students can uncover the connections between mathematics and other branches of knowledge.

Figure 4. Images of objects showing at least one symmetry
Source: Photographs by the authors.

Another geometric element that, although relatively recent, can be introduced in elementary school mathematics and is closely related to the arts and sciences – from medicine to economics, and from furniture design to painting – is the concept of fractals. These elements belong to fractal geometry, a field developed in 1975 by the French mathematician Benoit Mandelbrot (1924 – 2010), whose work was greatly facilitated by advances in computer technology. Figure 5 shows the creator alongside one of his fractals.

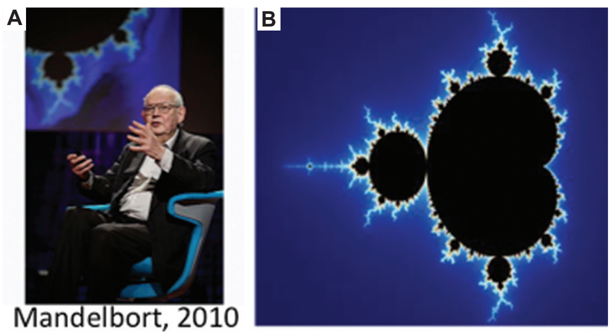


Figure 5. A photograph of Mandelbrot (A) and one of his fractals (B)
Source: Images from Wikipedia.

Mandelbrot was one of the first to use computer graphics to create and display fractal geometric images, leading to his discovery that visual complexity can arise from simple rules. The equations governing these fractals describe phenomena that, while appearing random, follow specific rules and cannot be explained by Euclidean geometry due to their fractional dimension. Fractals possess two main characteristics: self-similarity, where each part of the fractal resembles the whole, and infinite complexity, where zooming in on a detail reveals new intricacies. In this way, fractals contribute to the creation of new shapes and more realistic artificial worlds, as well as the representation of natural elements that Euclidean geometry cannot capture. They are of great importance, for example, in the creation of special effects in films. An example of a fractal in nature is the Romanesco Broccoli, as shown in Figure 6.

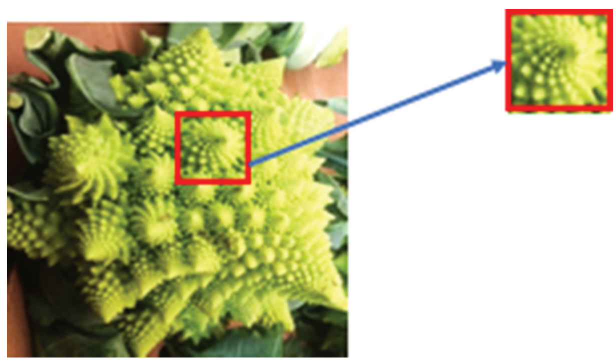


Figure 6. Image of a *Romanesco Broccoli*
Source: Photograph by the authors.

The concept of fractal is widely used in different forms of artistic production, including those created through computer animation, as illustrated in Figure 7.

To conclude this journey through the history of artists who used mathematics to create their works – works that can serve as a powerful resource for mathematics education from elementary school onward – we would like to highlight a concept that goes by several names (e.g., Golden Rectangle, Golden Number, Golden Ratio, Golden Section, Golden Proportion): golden number, whose history is lost in the depths of time. While it was used in ancient Egypt and the Pythagorean tradition, the first formal definition is attributed to Euclid (325 – 265 BC). The first treatise on this topic, entitled "Divina Proportione", was written by Luca Pacioli (1445 – 1517) and illustrated by Leonardo Da Vinci, to whom, according to tradition, the name Golden Section is credited.[8]

The Italian Leonardo da Vinci (1452 – 1519) was one of the painters most concerned with the use of the golden number, or divine proportion, in his works. Among the

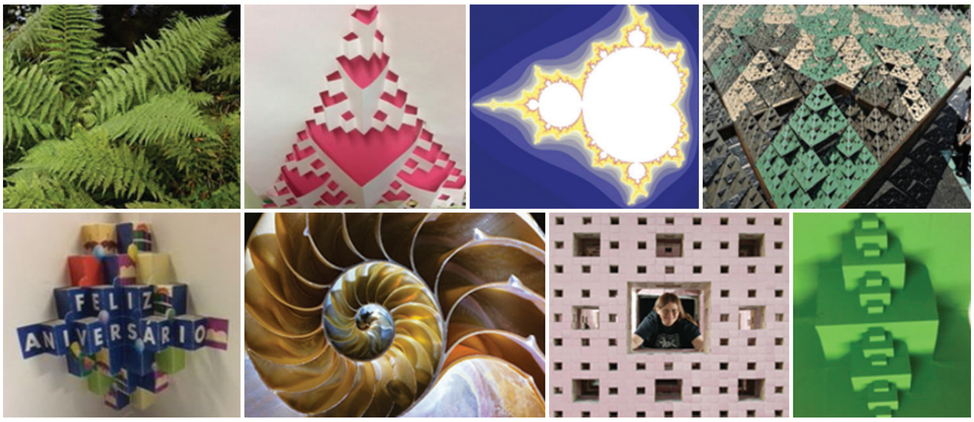


Figure 7. Examples of fractals in various contexts
Source: Images from Getty Images and the authors.

most emblematic creations is the Mona Lisa (Figure 8), which contains several examples of these ratios. Another example is the Vitruvian Man, where, for instance, the ratio between the height (side of the square) and the distance from the navel to the tip of the hand (radius of the circle) corresponds to the golden ratio. Another notable figure is Mondrian, who will be discussed in more detail later in this article. Along with da Vinci, Mondrian is among the most famous artists who applied the golden ratio into their works. Georges Seurat (1859 – 1891) is another example, with his painting "A Bath in Asnières", which shows obvious subdivisions based on the golden rectangle (Figure 8).

This number can be found in works spanning the Renaissance period to 21st-century modern art. The golden number is often regarded as the mathematical representative of perfection in nature, which is why, over 2000 years ago, humanity recognized its potential to enhance art and architecture. Its harmonious proportions are particularly pleasing to the eye. Studied since antiquity, the golden ratio forms the basis of many Greek constructions and artistic creations. The Parthenon in Athens is perhaps the best example of the intersection of mathematics and art. The name adopted for the golden number – capital Phi – is derived from the name of Phidias, the sculptor and architect responsible for the construction of the Parthenon in Athens. In Figure 9**,** some proportions on the façade of the Parthenon can be seen. However, this ratio can also be found in Egypt on the Pyramid of Giza, as well as in the façade of the Taj Mahal in India.

Similarly, the works of the Franco-Swiss architect, urban planner, painter, and sculptor Le Corbusier (1887 – 1965) also make use of the Golden Rectangle or the Golden Number. Figure 10 illustrates some of his creations.

The golden rectangle, known for its visually pleasing proportions, is not only employed in art and architecture

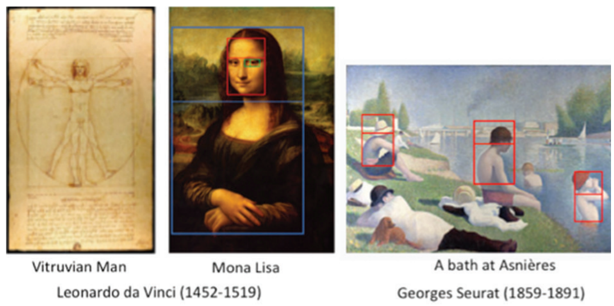


Figure 8. Examples of paintings where the golden rectangle is visible
Source: Illustrations from Pixabay and the authors.

but also found in everyday objects such as credit cards, identification cards, playing cards, books, and notepads, among others. A simple method of constructing a golden rectangle is illustrated in Figure 11.

4. Exploring mathematics through artistic productions

In this section, we present examples of tasks designed for future elementary school teachers to foster connections between mathematics and art. These tasks also aim to raise future teachers' awareness of art, thereby broadening their cultural repertoire, which is often limited and not fully appreciated. In other words, the goal is to use artistic works in mathematics classes not just as illustrations or props to support a particular task or piece of information, but as objects for learning mathematics, revealing the underlying mathematical principles. Working with shapes involves identifying part-to-whole relationships, which in turn allows us to observe that in a different object, understood as a whole, there is a structure of similarity and transformation.[15] On the other hand, national and international recommendations encourage interdisciplinarity through activities encompassing

Figure 9. The golden ratio in ancient architecture
Source: Images and illustration from Getty Images (Párthenon) and the authors (the Great Pyramid of Giza and Taj Mahal).

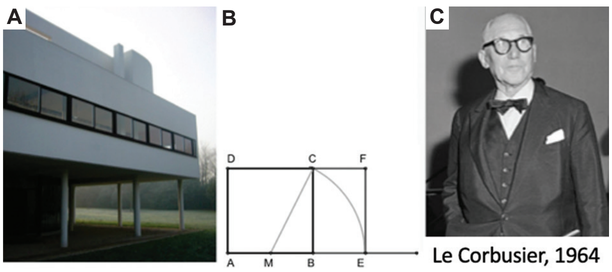


Figure 10. A photograph of the building by the architect Le Corbusier (A), the golden rectangle (B) and a photograph of Le Corbusier (C)
Source: Images and illustration from Roryrory (the building "Le Corbusier-Villa Savoye, 1928-30"; licensed under CC BY-SA 2.0) (A), the authors (B), and Wikipedia (C).

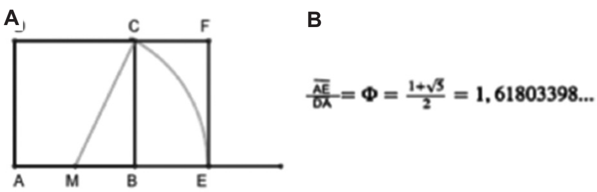


Figure 11. Construction of a golden rectangle. (A) Draw a square ABCD. Extend the segment AB, one side of the square, and find the point of intersection, E, of that segment with the arc of a circle whose radius is the segment MC, and center at M, the midpoint of the side of the square AB. Draw a perpendicular to AE through the point E, obtaining EF. Complete the rectangle by drawing a parallel AE through F. Obtain the constructed rectangle AEFD which is a golden rectangle. (B) The ratio between the dimensions of the constructed rectangle (golden ratio) is equal to the golden number Phi which is an irrational number.
Source: Illustration from the authors.

Science, Technology, Engineering, Art, and Mathematics (STEAM) education or mathematical connections within (e.g., mathematics, geometry) and beyond mathematics (e.g., real-life applications). This approach seeks to break the compartmentalized and isolated view of mathematics, promoting its integration with other fields.[2-5]

These experiences, which underscore the connections between mathematics and art, hold added value from an educational standpoint. In this perspective, we have sought to underscore this potential within the context of initial teacher training by integrating proposals and tasks that future teachers may employ with their own students. The tasks devised for this purpose have been designed to encompass a wide range of dimensions that are not typically addressed in other proposals, including hands-on activities, as well as the artistic and cultural domains. These tasks have been incorporated into training programs for elementary school teachers, specifically prospective teachers for pupils aged 3 – 12, within the domain of didactics of mathematics. They are closely aligned with the respective syllabus, particularly in geometry or algebra modules, as well as more transversal modules, such as those addressing mathematical connections and creativity. As a result, these tasks naturally arise in the classroom and are implemented based on the content covered at the time (e.g., sequences, geometric transformations, properties of polygons, fractions). The tasks are presented in a contextualized manner, thereby reinforcing their intentionality. Then, they are explored and discussed to ensure that future teachers develop a deep understanding of how to implement these strategies in the classroom. The future teachers' reactions are predominantly positive, acknowledging the value of these approaches, perceiving them as being distinct from conventional methods and as promoting enhanced motivation and meaningful learning.

4.1. Task 1: The fractal carpet

This task[16] is based on a wooden panel (Figure 12A) that uses the properties of a fractal, in particular the Sierpinski triangle (Figure 12B). It serves as a rich example to explore in a mathematics class, providing opportunities to formulate a variety of mathematical problems involving several concepts.

An artistic production, such as a panel, can lead to a rich mathematical exploration. The main mathematical idea is the concept of pattern, which involves identifying its formation that allows it to continue, and to which corresponds a structure translatable by a mathematical law. The discovery of patterns enables the establishment of connections between various mathematical topics in both numerical and geometric contexts. In this case, the first idea is to discover the pattern used in the construction of the carpet design. The visible diagonal of the square that constitutes the carpet divides it into two triangles with two different types of patterns, each obeying the same law of formation. The part formed by the triangles corresponds to

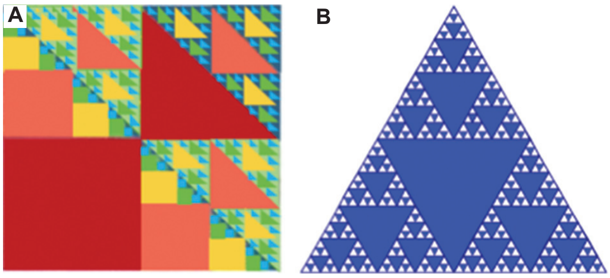
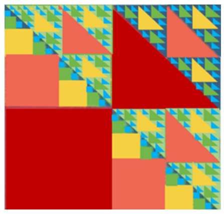


Figure 12. The fractal panel (A) and the Sierpinski triangle (B)
Source: Illustrations from the authors.

Task: The carpet fractal
Inês has the carpet shown in the picture at home.
One day she looked at it closely and discovered how its design was constructed. Can you do it too?



Classify all the polygons you see in the picture. Focus on the yellow squares.
The region formed by these squares, which part of the carpet is it?

Now focus on the triangles. What are the measures of the angles of one of the triangles? What about the others? What is the relationship between the yellow triangle and the orange triangle? What color are the triangles that together make up 3/32 of the carpet?

What percentage of the carpet is painted in red? What fraction of the original square is represented by the red square? Indicate the area of the red square, taking the original square as the unit area.

Now use the fractal principle identified in the image to make your masterpiece.

Figure 13. The carpet fractal task
Source: Illustration from the authors.

the Sierpinski Triangle structure (Figure 13), a geometric figure obtained through a recursive process. This figure belongs to a class of mathematical objects called fractals, characterized by self-similarity, meaning they do not lose their initial definition when they are enlarged (or reduced). Upon closer examination, we can unveil other mathematical concepts that may not be immediately apparent, such as fractions, ratios, proportions, powers, percentages, and areas. Hence, based on Figure 12, we propose the task described in Figure 13, which is suitable for primary education.

4.2. Task 2: Now you are the artist Mondrian

The works of the Dutch painter Piet Mondrian (1872 – 1944), particularly "Composition with Red," "Yellow and Blue" (1942), "Composition with Large Red Plane, Yellow, Black, Gray and Blue" (1921), and "Composition A" (1920) (Figure 14), provide a strong link between art and mathematics. This association can be used to formulate mathematical tasks to explore with students

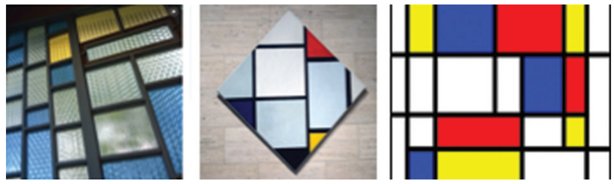


Figure 14. Photographs of Mondrian's paintings
Source: Images from Pixabay.

from elementary school onward, allowing them to get to know the work of an artist who uniquely incorporates basic elements of geometry.[17,18]

Mondrian's process involves simplifying forms to achieve a composition governed by ideal mathematical proportions, such as the golden ratio, which dictates the relationships between the forms. According to Mondrian, each thing, whether a house, a tree, or a landscape, has an essence that lies beyond its appearance, and in this essence, all things exist in harmony within the universe. The artist's role, as perceived by Mondrian, is to reveal this hidden essence and its connection to universal harmony. He developed a completely non-objective style known as Neoplasticism. Neoplasticism, also known as the De Stijl Movement (or "The Style"), introduced a transformative esthetic rooted in formal purification. It advocated a total spatial cleansing of painting, reducing it to its purest elements and seeking out its most characteristic features.[17-19]

Mondrian believed in a deep connection between mathematics and art. He used the simplest geometric shapes – straight lines, rectangles - and primary colors – blue, red, yellow – in their most saturated, artificial forms, alongside white, black, and gray. These latter tones, absent in nature (with white representing the total presence of light and black the total absence), were used to express reality, nature, and logic from a unique point of view. His lines were always straight, horizontal, or vertical, avoiding angles other than 90º or curved lines. Mondrian's point of view rooted in the idea that any shape can be created with basic geometric forms, just as any color can be created with different combinations of red, blue, and yellow. The golden rectangle is one of the basic shapes that is consistent in Mondrian's art.[19]

Figure 15 describes a task proposed by a group of initial teacher training students in the context of creating mathematical problems. As part of the activity, they were asked to formulate a set of questions inspired by a Mondrian painting that could be applied in early years mathematics class. In addition, the task was contextualized by exploring Mondrian's work and his contribution to the visual arts.

4.3. Task 3: Friezes and rosettes

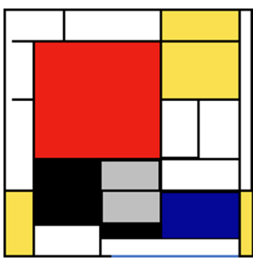
The tasks involve geometric transformations, and isometries reflection, rotation, translation, and glide reflection – with a particular emphasis on symmetries. These tasks are playful in nature, as they can be explored through concrete situations in decorative art.

The tasks in this area seek to uncover connections between mathematics and art through artistic creations in various forms. These artistic expressions achieve their pinnacle in religious art, especially in convents and churches, whether in tiles, stained glass, or stonework, as well as in handicrafts, such as basketry and embroidery – made by simple craftsmen or more erudite artists. The highlighted mathematical elements are friezes and rosettes, which exhibit patterns that acquire a specific meaning here – different from that considered in the numerical field – as it is characterized by the identification of a repeating motif.[13] The law of formation underlying the structure of the pattern makes it possible to understand how they were constructed, which allows them to be continued or reproduced mathematically.

The idea behind these geometric elements is the concept of symmetry, a powerful idea not only in geometry but also in art because of the balance and harmony it conveys. Mathematically, symmetry refers to the properties of a figure, which is defined as a subset of points in a plane or space, depending on the context. Hence, we can discuss the symmetry, or symmetries, of various objects, such as a straight line, rectangle, sphere, or even an artistic drawing or sculpture, as long as they are understood as subsets of points within a plane or space.[12] Based on these symmetries, figures can be classified into different types. In this section, we will focus on rose windows and friezes.

Task: Now you are the artist Mondrian

Observe the given picture based on Mondrian's compositions.



What can you say about the picture? What types of quadrilaterals can you identify in the picture? Find all the squares.

Cut out all the pieces that make up the picture. Find out which pieces are 1/4 of the red square.

Which part of the black square is one of the grey rectangles?

Find out which pieces are twice the size of the black rectangle. Which pieces are 1/8 of the red square? Which pieces are 1/2 of the yellow square?

The board is approximately 95 cm in length. Estimate the area of the red square. Estimate the perimeter of the black square.

Check whether the rectangle formed by the red board, black square, and two grey rectangles is a golden rectangle. Investigate other rectangles.

Use the pieces you've cut out and build a new picture that has at least one reflection symmetry.

Look at the picture you've built. Can you identify a rotational symmetry?

Now you're the artist. Imagine you're Mondrian and use his technique identified in the painting to make your masterpiece.

Figure 15. "Now you are the artist Mondrian" task
Source: Illustration from the authors.

The image on the left (Figure 16) does not have an axis of symmetry, as there is no straight line in the plane that acts as the axis of a reflection to leave the figure invariant. In other words, it has no reflection symmetry. However, it does have rotational symmetry, specifically six rotational symmetries; by rotating the figure in the plane around its center (O) at angles of 60°, 120°, 180°, 240°, 300°, and 360°, the figure remains unchanged. The image on the right (Figure 16) has both reflection symmetry (with four axes of reflection) and rotational symmetry (with four rotational symmetries). In both cases, the figures are examples of rosettes, where several geometrically equal modules are identified and repeated.

Rosette windows were an important element in Gothic architecture, commonly used on the façades of many churches. They were also used in Romanesque architecture, which also made use of these elements, particularly on the sides of buildings. Figure 17 illustrates two types of rosette windows: the one on the left was obtained by folding paper and the ones in the middle and right are part of the Basilica of Santa Luzia at Viana do Castelo, shown from both the exterior and interior of the basilica.

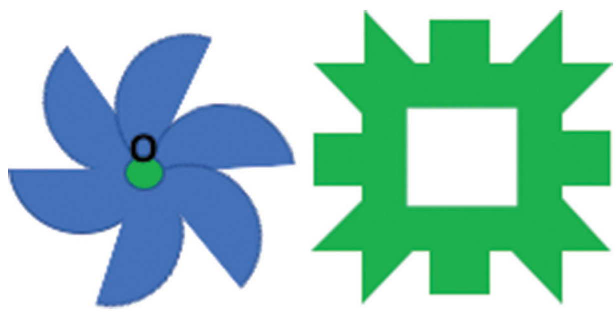


Figure 16. Examples of rosettes
Source: Illustrations from the authors

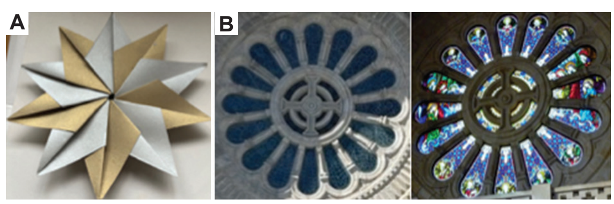


Figure 17. Rosette in a folded paper (A) and in Basilica of Santa Luzia at Viana do Castelo (B)
Source: Photographs from the authors.

It is important to highlight the nature of mathematical elements, particularly geometric elements, which are typically colorless or monochromatic. Color can have a significant influence in the identification of symmetries.[12] Thus, the mathematical study of symmetry is often conducted with monochromatic figures. However, most artistic objects that captivate us are polychromatic, making their analysis more complex, especially when working with younger students. For example, in the paper rosette in Figure 17, it is not easy to determine whether a rotation centered at O with an amplitude of 360º qualifies as a symmetry of the figure. To address this, such figures should either be avoided or analyzed under specific conditions to simplify their study.

Let us now consider Figure 18, which consists of 18th-century (Baroque) tiles commonly found in many churches and some public buildings, such as the Cloister of Nª Srª do Carmo in Viana do Castelo. Imagine that this pattern extends indefinitely in both directions, with a motif identified that repeats endlessly. We say that it has translation symmetry because, by translating the plane according to the vector AB, the entire figure is transformed into itself, although no point in the figure remains invariant under this transformation. A translation according to the vector BA is also a symmetry of the figure, as are all translations along multiple vectors. Tracing paper is a helpful material for identifying such transformations.

In this way, we have a frieze, which is a band with a motif that repeats indefinitely, characterized by translational symmetries in a single direction. It can also possess additional symmetries beyond translation. Depending on the symmetries that the friezes can have, they are classified into seven types. For example, the frieze in Figure 18 not only has translation symmetry but also sliding reflection symmetry. To illustrate, imagine one of the “Cs” (a simple motif) that is reflected by a sliding reflection along a horizontal axis, which crosses the band in the middle, and a horizontal vector equal in length to the distance between the two “Cs”. The initial “C” and the final “C”, obtained by a sliding reflection, make up the composite motif which, by translation, constitutes the frieze.

To solve the following task illustrated in Figure 19, students must apply the concept of symmetry while keeping in mind the properties of geometric figures, such as regular polygons, circumferences, and angles. These tasks encourage creativity and provide an opportunity to relate mathematics to real life.

Figures 20 and 21 illustrate some of the creations of the students, future elementary school teachers, and kindergarten teachers, who were challenged to design a bookmark and a glass coaster as examples of a frieze and a rosette, respectively. Each of the productions, presented by these students, was accompanied by photographs showing the entire process used to build their productions, as well as the mathematical explanation involved in both the friezes and the rosettes.

5. Conclusion

Learning mathematics in a fragmented and isolated way from grades 1 – 12 will not adequately prepare future citizens for the challenges ahead. Education in the 21st century has to break away from this dominant

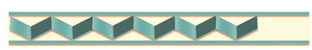
Figure 18. An example of a frieze (AB is one of the vectors that keep the translation invariant)
Source: Photograph from the authors.

Task: The rosette task

1. The picture on the right represents nine “windows” of various churches.
   a) Which window rosettes have reflection symmetry?
   b) Identify all the rotational symmetries in each of the windows
   c) Which regular polygons could you inscribe on each of the circumferences?

2. With all the elements on the left, build a figure that is a rosette, with two reflection symmetries and two rotational symmetries.

Task: The frieze task



Look at the drawing of a frieze in one of the rooms at the University of Alcalá, Spain.

1. What can you say about the picture? Summarize your findings. If necessary, use tracing paper to discover/confirm some of your findings.
2. What are the symmetries of the frieze that keep it globally invariant?
3. What was the minimum motif that generated the frieze?
4. How would you classify the geometric figures that make it up?
5. Imagine that the given figure is limited. Estimate the percentage of the part painted in green in relation to the whole.

Task: Be the artist

Create a bookmark (frieze) and a glass coaster (rosette).
You can use any materials you like. For each of the elements you build, clearly identify the type of frieze and rosette used and their properties.

Figure 19. Rosette and frieze tasks
Source: Images and illustration from the authors.

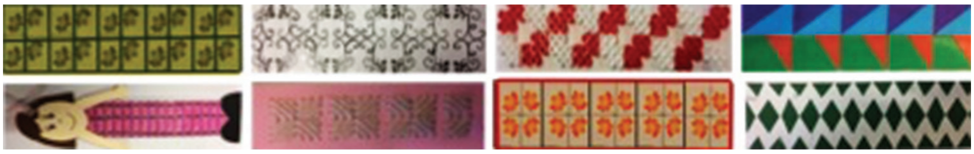


Figure 20. Bookmarks designed by students, future elementary school teachers, and kindergarten teachers
Source: Images from the authors.

Figure 21. Glass coasters designed by students, future elementary school teachers, and kindergarten teachers
Source: Photographs from the authors.

disciplinary paradigm, emphasizing connections and projects that are meaningful to students and aligned with teaching objectives. Education today needs to do far more than just transmit content; it must foster creativity, critical thinking, problem-solving, and decision-making skills. It should also help students improve their ability to communicate and collaborate effectively while developing their esthetic and artistic sensibilities.[20] Mathematics education should be contextualized, integrating knowledge from other domains and languages, allowing students the opportunity to be imaginative, to question, and to be curious. In this way, we contribute to the development of a more informed and cultured future citizen.

When students enter school, they already possess a strong potential, which teachers must recognize and make the most of. By doing so, young learners are more likely to enjoy mathematics and appreciate its usefulness, rather than applying a set of trained mathematical techniques. This approach makes mathematics a powerful tool in the society. In this sense, the teacher must be attentive and use different ways to explore students' potential, ensuring that each student's learning process is supported and understood.[16]

The analysis (and construction) of artistic objects through a mathematical lens offers opportunities to establish connections between mathematics and other domains. This task can serve as a starting point for projects where mathematics in general, and geometry, in particular, play a key role for those who want to learn mathematics and for those who want to develop harmonious artistic projects. Like other sciences, art can play a vital role in this process, with initial and ongoing teacher training programs being essential to its success.

In conclusion, a mathematically competent student, like an artist, can go beyond the use of mathematical elements in their daily work.[21] Drawing from the findings of our research in teacher training, we have seen that teaching mathematics in connection with visual arts and architecture offers several important benefits that enhance both subjects. This interdisciplinary approach fosters a deeper understanding of both subjects and promotes a more holistic educational experience. Furthermore, it provides future teachers with opportunities to engage with different artists and their creations – experiences they might not otherwise encounter – and to develop an interdisciplinary perspective on mathematics, along with all the benefits this approach offers.

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Conflict of interest

The authors declare no conflicts of interest.

Author contributions

Conceptualization: All authors
Formal analysis: All authors
Investigation: All authors
Methodology: All authors
Visualization: All authors
Writing – original draft: All authors
Writing – review & editing: All authors

Ethics approval and consent to participate

Not applicable.

Consent for publication

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Availability of data

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Further disclosure

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References

1. Kline M. *Mathematics in Western Culture*. United Kingdom: Oxford University Press; 1964.
2. Diego-Mantecón JM, Blanco TF, Búa Aresc JB, Sequeiros PG. Is the relationship between art and mathematics addressed thoroughly in Spanish secondary school textbooks? *J Math Arts*. 2019;13(1-2):25-47.

   doi: 10.1080/17513472.2018.1552068
3. ME-DGE. *Aprendizagens Essenciais: Matemática* [*Essential Learnings: Mathematics*]. Ministério da Educação/Direção-Geral da Educação; 2021. Available from: https://www.dge.mec.pt [Last accessed on 2024 Sep 30].
4. European Commission. *Proposal for a Council Recommendation on Key Competences for Lifelong Learning*. EU; 2018. Available from: https://ec.europa.eu/education/sites/education/files/recommendation-key-competences-lifelong-learning.pdf [Last accessed on 2024 Sep 15].
5. Vale I, Barbosa A, Peixoto, A, Fernandes, F. Solving problems through engineering design: An exploratory study with pre-service teachers. *Educ Sci*. 2022;12:889.

   doi: 10.3390/educsci12120889
6. Hardy GH. *A Mathematician's Apology*. University of Alberta Mathematical Sciences Society; 1940. Available from: https://web.njit.edu›~akansu›papers›ghhardy-amathematiciansapology.pdf [Last accessed on 2024 Jul 12].
7. Stix A. The Link between Art and Mathematics. In: *Proceedings of the Annual Conference of the National Middle School Association*. NMSA; 1995.
8. CMUP. *Arte e Matemática* [*Art and Mathematics*]. Centro de Matemática da Universidade do Porto; 2010. Available from: https://cmup.fc.up.pt/Steecmup/arte/plasticas/index.html [Last accessed on 2024 Sep 15].
9. Sawyer W. *Prelude to Mathematics*. United Kingdom: Penguin Books; 1955.
10. Steen L. The science of patterns. *Science*. 1988;240:611-616.

    doi: 10.1126/science.240.4852.611
11. Durer A. *The Complete Engravings, Etchings and Drypoints of Albrecht of Dürer*. United States: Dover Publications; 2000.
12. Bastos R. Notas sobre o ensino da geometria do grupo de trabalho de geometria da APM - simetria [Notes about the teaching of geometry of the group work of geometry of APM - symmetry]. *Educ Mat*. 2006;88:9-11.
13. Veloso E. *Simetrias e Transformações Geométricas* [*Symmetry and Geometric Transformations*]. Lisboa: APM; 2012.
14. Serra M. *Discovering Geometry - An Investigative Approach*. California: Key Curriculum Press; 2008.
15. Gimenez J. *La proporción: Arte y Matemáticas* [*The Proportion: Art and Mathematics*]. Barcelona: Editorial Graó; 2009.
16. Vale I, Pimentel T. Padrões e conexões matemáticas no ensino básico [Patterns and connections in elementary education]. *Educ Mat*. 2010;110,33-38.
17. Sylviani S, Permana FC, Azizan AT. Enhancing mathematical interest through visual arts integration: A systematic literature review. *Int J Educ Math Sci Technol*. 2024;12(5):1217-1235.

    doi: 10.46328/ijemst.4118
18. Robbins D. *Cubism and Abstract Art: Piet Mondrian. Guggenheim Museum Archives Reel-to-Reel Collection - The Solomon R. Guggenheim Foundation* (*SRGF*); 1962 Available from: https://www.guggenheim.org›wp-content›uploads›2018›08›615214t29_01_piet-mondrian [Last accessed on 2024 Jul 20].
19. Schapiro M. *Mondrian - A Dimensão Humana da Pintura Abstrata [On the Humanity of Abstract Painting]*. Brazil: Cosac and Naify; 2001.
20. Schleicher A. *Teaching Excellence through Professional Learning and Policy Reform: Lessons from Around the World*. Paris, France: OECD; 2016.

    doi: 10.1787/9789264252059-en
21. Vale I. Matemática e arte: Uma conexão a explorar no ensino da matemática [Mathematics and art: A connection to explore in mathematics teaching]. *Diálogos Arte*. 2017;7:223-242.